Error Measurements

Let v_A be the approximate value and v_E be the exact value

- Absolute error: $|v_A - v_E|$
- Relative error: $\frac{|v_A - v_E|}{v_E}$
- Percentage error: $\frac{|v_A - v_E|}{v_E} \times 100\%$

Taylor Series

Familiar (and useful) examples of Taylor series are the following:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad (|x| < \infty) \tag{1}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \qquad (|x| < \infty) \tag{2}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \qquad (|x| < \infty) \tag{3}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{k=0}^{\infty} x^k \qquad (|x| < 1) \tag{4}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k} \qquad (-1 < x \leqq 1) \tag{5}$$

Taylor Series about a point

1.2 Taylor Polynomials

One of the cornerstones of numerical analysis is Taylor's theorem about which you learned in Calculus. A short study bears repeating here, however.

Theorem 1. *Suppose that $f(x)$ has $n+1$ derivatives on (a, b), and $x_0 \in (a, b)$. Then for each $x \in (a, b)$, there exists a ξ, depending on x, lying strictly between x and x_0 such that*

$$f(x) = f(x_0) + \sum_{j=1}^{n} \left(\frac{f^{(j)}(x_0)}{j!} (x - x_0)^j \right) + \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)^{n+1}.$$

Pseudo-code for Bisection

Step 1: Set $L = f(a)$;

Step 2: Set $m = \frac{a+b}{2}$; $M = f(m)$;

Step 3: If $LM < 0$ then set $b = m$; else set $a = m$ and $L = M$;

Step 4: Go to Step 2.

False Position

$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$$

Newton's Method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Secant Method

$$x_{n+1} = x_n - g(x_n)\frac{x_n - x_{n-1}}{g(x_n) - g(x_{n-1})}$$

Trapezoid Rule

$$I = (b-a)\frac{f(a) + f(b)}{2}$$

Trapezoid Rule, Uniform Spacing

$$\int_a^b = f(x)dx \approx T(f;P) = h\left\{\sum_{i=1}^{n-1} f\left(x_i\right) + \frac{1}{2}\left[f\left(x_0\right) + f\left(x_n\right)\right]\right\}$$

Simpson's 1/3 Rule

$$I = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]$$

where $h = (b-a)/2$

- The formula is

$$I \approx (b-a)\ \frac{f(x_0) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{j=2,4,6}^{n-2} f(x_j) + f(x_n)}{3n}$$

where $h = \frac{b-a}{n}$

Definition of Simpson's 3/8 Rule

$$I = \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)]$$

where $h = (b-a)/3$

Romberg Integration

[redacted]

```
procedure Romberg(f, a, b, n, (r_ij))
real array (r_ij)_{0:n×0:n}
integer i, j, k, n

real a, b, h, sum
interface external function f
h ← b − a
r_00 ← (h/2)[f(a) + f(b)]
for i = 1 to n do
    h ← h/2
    sum ← 0
    for k = 1 to 2^i − 1 step 2 do
        sum ← sum + f(a + kh)
    end for
    r_i0 ← (1/2) r_{i−1,0} + (sum)h
    for j = 1 to i do
        r_ij ← r_{i,j−1} + (r_{i,j−1} − r_{i−1,j−1})/(4^j − 1)
    end for
end for
end procedure Romberg
```

Gauss's Formula

- $\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i)$
- Note that x_i and w_i are the points and weights of Gaussian quadrature

Gauss's Formula, arbitrary finterval

- $\int_{a}^{b} f(y)dy \approx \frac{b-a}{2} \sum_{i=1}^{n} w_i f(y_i)$
- $y_i = \left(\frac{b-a}{2}\right) x_i + \left(\frac{b+a}{2}\right)$
- Note that x_i and w_i are the points and weights of Gaussian quadrature