Section 1

MANE 3332.03

Subsection 1

Chapter 4 Content

Continuous Random Variable

- The **probability distribution** of a random variable X is a description of the set of probabilities associated with the possible values of X
- Density functions are commonly used in engineering to describe physical systems. (if you measure it it is real)
- A **probability density function** $f(x)$ can be used to describe the probability distribution of a continuous random variable pdf

Probability Density Function

- Notice the difference from a discrete random variable
- The formal definition of a probability density function is a function such that
  1. $f(x) \geq 0$
  2. $\int_{-\infty}^{\infty} f(x)\, dx = 1$
  3. $P(a \leq X \leq b) = \int_{a}^{b} f(x)\, dx$

Chapter 3

(1) the same

(2) $\sum_{all\ x} f(x) = 1$

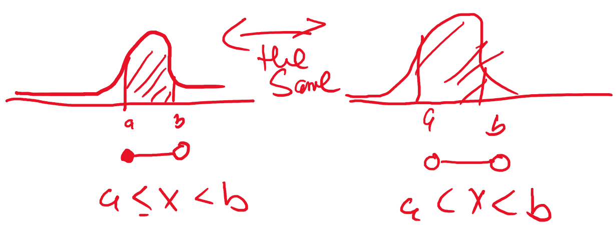
not ~~chapter 3~~

Probability Density Function

- Any interesting property of continuous random variables is

$$P(x_1 \leq X \leq x_2) = P(x_1 < X \leq x_2) = P(x_1 \leq X < x_2)$$
$$= P(x_1 < X < x_2)$$

- Does not apply to discrete random variables
- Explanation



left figure differs by adding the area when X=a exactly

Cumulative Distribution Function

The cumulative distribution function for a continuous random variable X is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(y)\, dy$$

Mean and Variance of a Continuous Random Variable

- The mean value of a continuous random variable is defined to be

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

- The variance of a continuous random variable is defined to be

$$\begin{aligned} \sigma^2 = & \quad V(X) = E(X-\mu)^2 \\ = & \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2 \end{aligned}$$

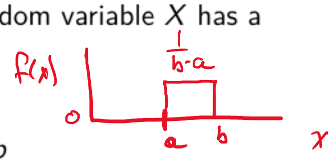
- The standard deviation of X is

$$\sigma = \sqrt{V(X)}$$

Continuous Uniform Distribution

The continuous uniform distribution is the analog of the discrete uniform distribution in that all outcomes are equally likely to occur

- A continuous uniform distribution for the random variable X has a probability density function



$$f(x) = \frac{1}{b-a}, \quad a \le x \le b$$

- The mean of the uniform distribution is

$$\mu = E(X) = \frac{a+b}{2}$$

$$\mu = \int_{-\infty}^{\infty} x \frac{1}{b-a}\, dx$$

$$= \int_{-\infty}^{a} x \frac{1}{b-a}\, dx + \int_{a}^{b} x\left(\frac{1}{b-a}\right) dx + \int_{b}^{\infty} x\left(\frac{1}{b-a}\right) dx$$

$$0 \qquad 0$$

- The variance of X is

$$\sigma^2 = V(X) = \frac{(b-a)^2}{12}$$

$$\int_{a}^{b} \frac{1}{b-a} x\, dx = \frac{1}{b-a} \int_{a}^{b} x\, dx$$

$$= \frac{1}{b-a} \left(\frac{x^2}{2}\right) \Big|_{x=a}^{b}$$

$$= \frac{1}{b-a} \left(\frac{b^2-a^2}{2}\right)$$

$f(x) = 2$ for $49.75 \le x \le 50.25$

$P(X < 50.1) = \int_{49.75}^{50.1} f(x)dx = \int_{49.75}^{50.1} 2dx = 2x\Big|_{x=49.75}^{50.1}$

Uniform Problem 4.1.6

- See page P-25

$= 2(50.1 - 49.75) = \underline{.70}$

$P(X > 49.9) = \int_{49.9}^{50.25} 2dx = 2x\Big|_{x=49.9}^{50.25} = \underline{.70}$

$P(X > 49.9) = 1 - F(49.9) = 1 - \frac{49.9 - 49.75}{50.25 - 49.75}$

$= 1 - .3 = \underline{.70}$

Attendance 1-A

Value of distribution s.t. $X < x = .90$

Gaussian

The Normal Distribution

- The normal distribution is the most widely used and important distribution in statistics.
- You must master this!
- A random variable X with probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \text{ for } -\infty < x < \infty$$

has a **normal distribution** with parameters μ and σ where $-\infty < \mu < \infty$ and $\sigma > 0$

- The normal distribution with parameters μ and σ is denoted $N(\mu, \sigma^2)$
- An interesting web-site is http://www.seeingstatistics.com/seeingTour/normal/shape3.html

Normal dist. is symmetric about μ

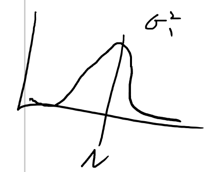
Mean and Variance of the Normal Distribution

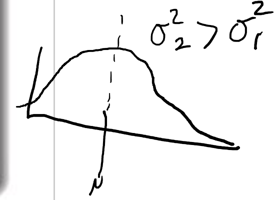
- The mean of the normal distribution with parameters μ and σ is

$$E(X) = \mu$$

- The variance of the normal distribution with parameters μ and σ is

$$V(X) = \sigma^2$$





Central Limit Theorem

- Brief introduction
- States that the distribution of the average of independent random variables will tend towards a normal distribution as n gets large n around 30
- More details later

1) can not integrate
2) Tables
3) Computer

$F(x) = \frac{x-a}{b-a}$

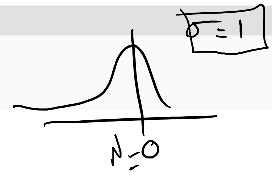
$\int_a^x f(x)dx$

Calculating Normal Probabilities

- Is somewhat complicated
- The difficulty is $\int_a^b f(x)\, dx$ does not have a closed form solution
- Probabilities must be found by numerical techniques (tabled values)
- It is very helpful to draw a sketch of the desired probabilities (I require this)

Strongly Suggest

The Standard Normal Distribution



- A normal random variable with $\mu = 0$ and $\sigma = 1$ is called a **standard normal** random variable
- A standard normal random variable is denoted as z
- The cumulative distribution function for a standard normal is defined to be the function

Φ – capital phi

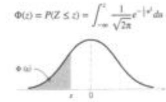
$$F(z) = \Phi(z) = P(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}$$

- These probabilities are contained in Appendix Table III on pages A-8 and A-9

Cumulative Standard Normal Distribution

A-8 APPENDIX A Statistical Tables and Charts

$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\,du$$



$P(z < -4.0) < P(Z < -3.99) = .00003 3$

for any $z < -3.99$
$P(Z < z) = 0.0$

$-3.99 < Z < 3.99$

TABLE III Cumulative Standard Normal Distribution

z	−0.09	−0.08	−0.07	−0.06	−0.05	−0.04	−0.03	−0.02	−0.01	−0.00
−3.9	0.000033	0.000034	0.000036	0.000037	0.000039	0.000041	0.000042	0.000044	0.000046	0.000048
−3.8	0.000050	0.000052	0.000054	0.000057	0.000059	0.000062	0.000064	0.000067	0.000069	0.000072
−3.7	0.000075	0.000078	0.000082	0.000085	0.000088	0.000092	0.000096	0.000100	0.000104	0.000108
−3.6	0.000112	0.000117	0.000121	0.000126	0.000131	0.000136	0.000142	0.000147	0.000153	0.000159
−3.5	0.000165	0.000172	0.000179	0.000185	0.000193	0.000200	0.000208	0.000216	0.000224	0.000233
−3.4	0.000242	0.000251	0.000260	0.000270	0.000280	0.000291	0.000302	0.000313	0.000325	0.000337
−3.3	0.000350	0.000362	0.000376	0.000390	0.000404	0.000419	0.000434	0.000450	0.000467	0.000483
−3.2	0.000501	0.000519	0.000538	0.000557	0.000577	0.000598	0.000619	0.000641	0.000664	0.000687
−3.1	0.000711	0.000736	0.000762	0.000789	0.000816	0.000845	0.000874	0.000904	0.000935	0.000968
−3.0	0.001001	0.001035	0.001070	0.001107	0.001144	0.001183	0.001223	0.001264	0.001306	0.001350
−2.9	0.001395	0.001441	0.001489	0.001538	0.001589	0.001641	0.001695	0.001750	0.001807	0.001866
−2.8	0.001926	0.001988	0.002052	0.002118	0.002186	0.002256	0.002327	0.002401	0.002477	0.002555
−2.7	0.002635	0.002718	0.002803	0.002890	0.002980	0.003072	0.003167	0.003264	0.003364	0.003467
−2.6	0.003573	0.003681	0.003793	0.003907	0.004025	0.004145	0.004269	0.004396	0.004527	0.004661
−2.5	0.004799	0.004940	0.005085	0.005234	0.005386	0.005543	0.005703	0.005868	0.006037	0.006210
−2.4	0.006387	0.006569	0.006756	0.006947	0.007143	0.007344	0.007549	0.007760	0.007976	0.008198
−2.3	0.008424	0.008656	0.008894	0.009137	0.009387	0.009642	0.009903	0.010170	0.010444	0.010724
−2.2	0.011011	0.011304	0.011604	0.011911	0.012224	0.012545	0.012874	0.013209	0.013553	0.013903
−2.1	0.014262	0.014629	0.015003	0.015386	0.015778	0.016177	0.016586	0.017003	0.017429	0.017864
−2.0	0.018309	0.018763	0.019226	0.019699	0.020182	0.020675	0.021178	0.021692	0.022216	0.022750
−1.9	0.023295	0.023852	0.024419	0.024998	0.025588	0.026190	0.026803	0.027429	0.028067	0.028717
−1.8	0.029379	0.030054	0.030742	0.031443	0.032157	0.032884	0.033625	0.034379	0.035148	0.035930
−1.7	0.036727	0.037538	0.038364	0.039204	0.040059	0.040929	0.041815	0.042716	0.043633	0.044565
−1.6	0.045514	0.046479	0.047460	0.048457	0.049471	0.050503	0.051551	0.052616	0.053699	0.054799
−1.5	0.055917	0.057053	0.058208	0.059380	0.060571	0.061780	0.063008	0.064256	0.065522	0.066807
−1.4	0.068112	0.069437	0.070781	0.072145	0.073529	0.074934	0.076359	0.077804	0.079270	0.080757
−1.3	0.082264	0.083793	0.085343	0.086915	0.088508	0.090123	0.091759	0.093418	0.095098	0.096801
−1.2	0.098525	0.100273	0.102042	0.103835	0.105650	0.107488	0.109349	0.111233	0.113140	0.115070
−1.1	0.117023	0.119000	0.121001	0.123024	0.125072	0.127143	0.129238	0.131357	0.133500	0.135666
−1.0	0.137857	0.140071	0.142310	0.144572	0.146859	0.149170	0.151505	0.153864	0.156248	0.158655
−0.9	0.161087	0.163543	0.166023	0.168528	0.171056	0.173609	0.176185	0.178786	0.181411	0.184060
−0.8	0.186733	0.189430	0.192150	0.194894	0.197662	0.200454	0.203269	0.206108	0.208970	0.211855
−0.7	0.214764	0.217695	0.220650	0.223627	0.226627	0.229650	0.232695	0.235762	0.238852	0.241964
−0.6	0.245097	0.248252	0.251429	0.254627	0.257846	0.261086	0.264347	0.267629	0.270931	0.274253
−0.5	0.277595	0.280957	0.284339	0.287740	0.291160	0.294599	0.298056	0.301532	0.305026	0.308538
−0.4	0.312067	0.315614	0.319178	0.322758	0.326355	0.329969	0.333598	0.337243	0.340903	0.344578
−0.3	0.348268	0.351973	0.355691	0.359424	0.363169	0.366928	0.370700	0.374484	0.378281	0.382089
−0.2	0.385908	0.389739	0.393580	0.397432	0.401294	0.405165	0.409046	0.412936	0.416834	0.420740

Cumulative Standard Normal Distribution

$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\, du$$

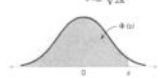
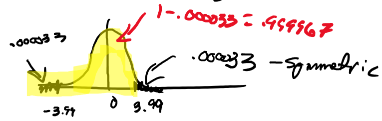


TABLE III Cumulative Standard Normal Distribution (*continued*)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.500000	0.503989	0.507978	0.511967	0.515953	0.519939	0.532922	0.527903	0.531881	0.535856
0.1	0.539828	0.543795	0.547758	0.551717	0.555760	0.559618	0.563559	0.567495	0.571424	0.575345
0.2	0.579260	0.583166	0.587064	0.590954	0.594835	0.598706	0.602568	0.606420	0.610261	0.614092
0.3	0.617911	0.621719	0.625516	0.629300	0.633072	0.636831	0.640576	0.644309	0.648027	0.651732
0.4	0.655422	0.659097	0.662757	0.666402	0.670031	0.673645	0.677242	0.680822	0.684386	0.687933
0.5	0.691462	0.694974	0.698468	0.701944	0.705401	0.708840	0.712260	0.715661	0.719043	0.722405
0.6	0.725747	0.729069	0.732371	0.735653	0.738914	0.742154	0.745373	0.748571	0.751748	0.754903
0.7	0.758036	0.761148	0.764238	0.767305	0.770350	0.773373	0.776373	0.779350	0.782305	0.785236
0.8	0.788145	0.791030	0.793892	0.796731	0.799546	0.802338	0.805106	0.807850	0.810570	0.813267
0.9	0.815940	0.818589	0.821214	0.823815	0.826391	0.828944	0.831472	0.833977	0.836457	0.838913
1.0	0.841345	0.843752	0.846136	0.848495	0.850830	0.853141	0.855428	0.857690	0.859929	0.862143
1.1	0.864334	0.866500	0.868643	0.870762	0.872857	0.874928	0.876976	0.878999	0.881000	0.882977
1.2	0.884930	0.886860	0.888767	0.890651	0.892512	0.894350	0.896165	0.897958	0.899727	0.901475
1.3	0.903199	0.904902	0.906582	0.908241	0.909877	0.911492	0.913085	0.914657	0.916207	0.917736
1.4	0.919243	0.920730	0.922196	0.923641	0.925066	0.926471	0.927855	0.929219	0.930563	0.931888
1.5	0.933193	0.934478	0.935744	0.936992	0.938220	0.939429	0.940620	0.941792	0.942947	0.944083
1.6	0.945201	0.946301	0.947384	0.948449	0.949497	0.950529	0.951543	0.952540	0.953521	0.954486
1.7	0.955435	0.956367	0.957284	0.958185	0.959071	0.959941	0.960796	0.961636	0.962462	0.963273
1.8	0.964070	0.964852	0.965621	0.966375	0.967116	0.967843	0.968557	0.969258	0.969946	0.970621
1.9	0.971283	0.971933	0.972571	0.973197	0.973810	0.974412	0.975002	0.975581	0.976148	0.976705
2.0	0.977250	0.977784	0.978308	0.978822	0.979325	0.979818	0.980301	0.980774	0.981237	0.981691
2.1	0.982136	0.982571	0.982997	0.983414	0.983823	0.984222	0.984614	0.984997	0.985371	0.985738
2.2	0.986097	0.986447	0.986791	0.987126	0.987455	0.987776	0.988089	0.988396	0.988696	0.988989
2.3	0.989276	0.989556	0.989830	0.990097	0.990358	0.990613	0.990863	0.991106	0.991344	0.991576
2.4	0.991802	0.992024	0.992240	0.992451	0.992656	0.992857	0.993053	0.993244	0.993431	0.993613
2.5	0.993790	0.993963	0.994132	0.994297	0.994457	0.994614	0.994766	0.994915	0.995060	0.995201
2.6	0.995339	0.995473	0.995604	0.995731	0.995855	0.995975	0.996093	0.996207	0.996319	0.996427
2.7	0.996533	0.996636	0.996736	0.996833	0.996928	0.997020	0.997110	0.997197	0.997282	0.997365
2.8	0.997445	0.997523	0.997599	0.997673	0.997744	0.997814	0.997882	0.997948	0.998012	0.998074
2.9	0.998134	0.998193	0.998250	0.998305	0.998359	0.998411	0.998462	0.998511	0.998559	0.998605
3.0	0.998650	0.998694	0.998736	0.998777	0.998817	0.998856	0.998893	0.998930	0.998965	0.998999
3.1	0.999032	0.999065	0.999096	0.999126	0.999155	0.999184	0.999211	0.999238	0.999264	0.999289
3.2	0.999313	0.999336	0.999359	0.999381	0.999402	0.999423	0.999443	0.999462	0.999481	0.999499
3.3	0.999517	0.999533	0.999550	0.999566	0.999581	0.999596	0.999610	0.999624	0.999638	0.999650
3.4	0.999663	0.999675	0.999687	0.999698	0.999709	0.999720	0.999730	0.999740	0.999749	0.999758
3.5	0.999767	0.999776	0.999784	0.999792	0.999800	0.999807	0.999815	0.999821	0.999828	0.999835
3.6	0.999841	0.999847	0.999853	0.999858	0.999864	0.999869	0.999874	0.999879	0.999883	0.999888
3.7	0.999892	0.999896	0.999900	0.999904	0.999908	0.999912	0.999915	0.999918	0.999922	0.999925

3.9 .999967

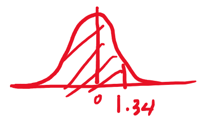
1-.000033 = .999967



6|.09

for P(Z > 3.99) = 1.0

Standard Normal Problem



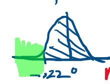
5.1.1 Suppose that $Z \sim N(0, 1)$. Find:

(a) $P(Z \leq 1.34)$
(b) $P(Z \geq -0.22)$
(c) $P(-2.19 \leq Z \leq 0.43)$
(d) $P(0.09 \leq Z \leq 1.76)$
(e) $P(|Z| \leq 0.38)$
(f) The value of x for which $P(Z \leq x) = 0.55$
(g) The value of x for which $P(Z \geq x) = 0.72$
(h) The value of x for which $P(|Z| \leq x) = 0.31$

Figure 3: image

$P(Z \leq 1.34) = \Phi(1.34)$
$= .909877$

Part b) $P(Z > -.22)$



$P(Z > -.22)$
$= 1 - \Phi(-.22)$
$= 1 - .412936$
$= .587064$

Standard Normal Practice Problems

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Standardizing (the z-transform)

$$z = \frac{x-\mu}{\sigma}$$

- Suppose X is a normal random variable with mean μ and variance σ^2

$$P(X \le x) = P\left(\overbrace{\frac{X-\mu}{\sigma}}^{Z} \le \frac{x-\mu}{\sigma}\right) = P(Z \le z)$$

- The z-value is $z = (x-\mu)/\sigma$
- Result allows the standard normal tables to be used to calculate probabilities for any normal distribution

Normal Probability Problem

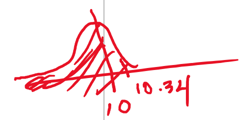
$\mu = 10, \sigma = 2$

5.1.3 Suppose that $X \sim N(10, 2)$. Find:

(a) $P(X \leq 10.34)$
(b) $P(X \geq 11.98)$
(c) $P(7.67 \leq X \leq 9.90)$
(d) $P(10.88 \leq X \leq 13.22)$
(e) $P(|X - 10| \leq 3)$
(f) The value of x for which $P(X \leq x) = 0.81$
(g) The value of x for which $P(X \geq x) = 0.04$
(h) The value of x for which $P(|X - 10| \geq x) = 0.63$

Figure 4: image

Part a



$P(X \leq 10.34) = P\left(Z \leq \frac{x-\mu}{\sigma}\right)$
$= P\left(Z < \frac{10.34 - 10}{2}\right)$
$= \Phi(.17)$
$= .567495$

Normal Practice Problems

$$Z = \frac{X - \mu}{\sigma}$$

naive approach → take mean & variance from binomial

Normal Approximation to the Binomial Distribution

- If X is a binomial random variable,

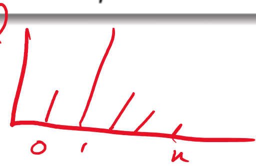
$$Z = \frac{X - np}{\sqrt{np(1-p)}}$$

is approximately a standard normal random variable. Consequently, probabilities computed from Z can be used to approximate probabilities for X

- Usually holds when

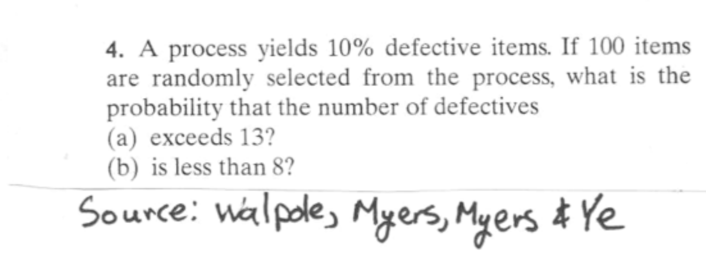
$$np > 5 \quad \text{and} \quad n(1-p) > 5$$

Binomial



Normal

Problem



4. A process yields 10% defective items. If 100 items are randomly selected from the process, what is the probability that the number of defectives

(a) exceeds 13?

(b) is less than 8?

Source: Walpole, Myers, Myers & Ye

Figure 5: image

- How good are the approximations?

Continuity Correction Factor

- Is a method to improve the accuracy of the normal approximation to the binomial
- Examine Figure 6.22 from Walpole, Myers, Myers & Ye. Note that each rectangle is centered at x and extends from $x - 0.5$ to $x + 0.5$
- This table should help formulate problems

Normal

Binomial Probability	with Correction Factor	Normal Approximation
$P(X \geq x)$	$P(X \geq x - 0.5)$	$P\left(Z > \frac{x-0.5-np}{\sqrt{np(1-p)}}\right)$
$P(X \leq x)$	$P(X \leq x + 0.5)$	$P\left(Z < \frac{x+0.5-np}{\sqrt{np(1-p)}}\right)$
$P(X = x)$	$P(x - 0.5 \leq X \leq x + 0.5)$	$P\left(\frac{x-0.5-np}{\sqrt{np(1-p)}} < Z < \frac{x+0.}{\sqrt{np(}}\right.$

Normal Approximation - Figure

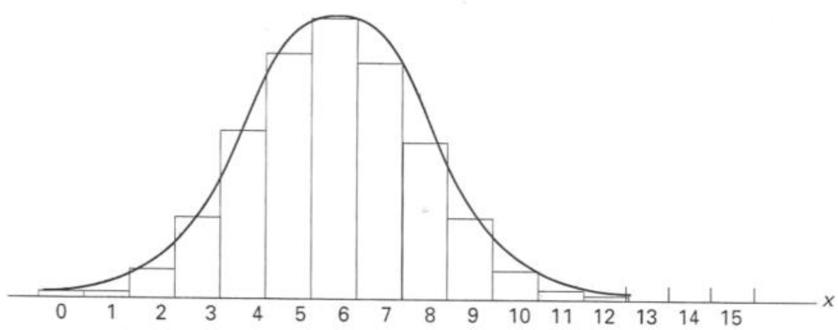


Figure 6.22 Normal approximation of $b(x; 15, 0.4)$.

Source: Walpole, Myers, Myers & Ye

Figure 6: image

Rework Problem using Continuity Correction Factor

- Are the approximations improved?

Normal Approximation to the Poisson Distribution

- If X is a Poisson random variable with $E(X) = \lambda$ and $V(X) = \lambda$,

$$Z = \frac{X - \lambda}{\sqrt{\lambda}}$$

is approximately a standard normal random variable.

Should include continuity correction

Exponential Distribution

- The exponential distribution is widely used in the area of reliability and life-test data.
- Ostle, et. al. (1996) list the following applications of the exponential distribution
  - the number of feet between two consecutive erroneous records on a computer tape,
  - the lifetime of a component of a particular device,
  - the length of a life of a radioactive material and
  - the time to the next customer service call at a service desk

Exponential Distribution

- The PDF for an exponential distribution with parameter $\lambda > 0$ is

$$f(x) = \lambda e^{-\lambda x}, \quad \text{for } 0 \leq x < \infty$$

- The mean of X is

$$\int_{-\infty}^{\infty} x f(x) dx \rightarrow \text{Integration by parts}$$

$$\mu = E(X) = \frac{1}{\lambda}$$

$$= \int_{0}^{\infty} (x) \lambda e^{-\lambda x} dx$$

- The variance of X is

Let $u = \lambda x \rightarrow \int_{0}^{\infty} u e^{-u} du$

$$\sigma^2 = V(X) = \frac{1}{\lambda^2}$$

inconvenience

Note that other authors define $f(x) = \frac{1}{\theta} e^{-x/\theta}$. Either definition is acceptable. However one must be aware of which definition is being used.

extra/extra careful with software

The Exponential CDF

The CDF for the exponential distribution is easy to derive

$$\begin{aligned}
F(x) = \quad P(X \le x) &= \int_{-\infty}^{x} \lambda e^{-\lambda y}\, dy \\
&= \int_{0}^{x} \lambda e^{-\lambda y}\, dy \\
&= \left(-e^{-\lambda y}\right)\Big|_{y=0}^{x} \\
&= -e^{-\lambda x} - (-e^{0}) \\
&= -e^{-\lambda x} + 1 \\
&= \boxed{1 - e^{-\lambda x}}
\end{aligned}$$

Problem 4-79

4-79. The time to failure (in hours) of fans in a personal computer can be modeled by an exponential distribution with $\lambda = 0.0003$.

(a) What proportion of the fans will last at least 10,000 hours?

(b) What proportion of the fans will last at most 7000 hours?

Figure 7: image

Lack of Memory Property

- The mathematical definition is

$$P(X < t_1 + t_2 | X > t_1) = P(X < t_2)$$

- That is "the probability of a failure time that is less than $t_1 + t_2$ given the failure time is greater than t_1 is the probability that the item's failure time is less than t_2
- This property is unique to the exponential distribution
- Often used to model the reliability of electronic components.

Problem 4–80

4-80. The time between the arrival of electronic messages at your computer is exponentially distributed with a mean of two hours.

(a) What is the probability that you do not receive a message during a two-hour period?

(b) If you have not had a message in the last four hours, what is the probability that you do not receive a message in the next two hours?

(c) What is the expected time between your fifth and sixth messages?

Figure 8: image

Relationship to the Poisson Distribution

- Let Y be a Poisson random variable with parameter λ. Note: Y represents the number of occurrences per unit
- Let X be a random variable that records the time between occurrences for the same process as Y
- X has an exponential distribution with parameter λ

Lognormal Distribution

- Let W have a normal distribution with mean θ and variance ω^2; then $X = \exp(W)$ is a **lognormal random variable** with pdf

$$f(x) = \frac{1}{x\omega\sqrt{2\pi}} \exp\left[-\frac{(\ln(x) - \theta)^2}{2\omega^2}\right] \quad 0 < x < \infty$$

- The mean of X is

$$E(X) = e^{\theta + \omega^2/2}$$

- The variance of X is

$$V(X) = e^{2\theta + \omega^2}\left(e^{\omega^2} - 1\right)$$

Example Problem

3-47. Suppose that X has a lognormal distribution with parameters $\theta = 5$ and $\omega^2 = 9$. Determine the following:

(a) $P(X < 13{,}300)$

(b) The value for x such that $P(X \leq x) = 0.95$

(c) The mean and variance of X Montgomery, Runger and Hubble

Figure 9: image

Gamma Distribution

- The random variable X with pdf

$$f(x) = \frac{\lambda^r x^{r-1} e^{-\lambda x}}{\Gamma(r)}, \quad \text{for } x > 0$$

is a **gamma random variable** with parameters $\lambda > 0$ and $r > 0$.

- The gamma function is

$$\Gamma(r) = \int_0^\infty x^{r-1} e^{-x}\, dx \quad \text{for } r > 0$$

with special properties:

- $\Gamma(r)$ is finite
- $\Gamma(r) = (r-1)\Gamma(r-1)$
- For any positive integer r, $\Gamma(r) = (r-1)!$

Gamma Distribution

- The mean and variance are

$$\mu = E(X) = r/\lambda \text{ and } \sigma^2 = V(X) = r/\lambda^2$$

- We will not work any probability problems using the gamma distribution

Gamma Tables

APPENDIX A

Gamma Function

n	$\Gamma(n)$	n	$\Gamma(n)$	n	$\Gamma(n)$	n	$\Gamma(n)$
0.0100	99.4327	0.5100	1.7384	1.0100	0.9943	1.5100	0.8866
0.0200	49.4423	0.5200	1.7058	1.0200	0.9888	1.5200	0.8870
0.0300	32.7850	0.5300	1.6747	1.0300	0.9836	1.5300	0.8876
0.0400	24.4610	0.5400	1.6448	1.0400	0.9784	1.5400	0.8882
0.0500	19.4701	0.5500	1.6161	1.0500	0.9735	1.5500	0.8889
0.0600	16.1457	0.5600	1.5886	1.0600	0.9687	1.5600	0.8896
0.0700	13.7736	0.5700	1.5623	1.0700	0.9642	1.5700	0.8905
0.0800	11.9966	0.5800	1.5369	1.0800	0.9597	1.5800	0.8914
0.0900	10.6162	0.5900	1.5126	1.0900	0.9555	1.5900	0.8924
0.1000	9.5135	0.6000	1.4892	1.1000	0.9513	1.6000	0.8935
0.1100	8.6127	0.6100	1.4667	1.1100	0.9474	1.6100	0.8947
0.1200	7.8632	0.6200	1.4450	1.1200	0.9436	1.6200	0.8959
0.1300	7.2302	0.6300	1.4242	1.1300	0.9399	1.6300	0.8972
0.1400	6.6887	0.6400	1.4041	1.1400	0.9364	1.6400	0.8986
0.1500	6.2203	0.6500	1.3848	1.1500	0.9330	1.6500	0.9001
0.1600	5.8113	0.6600	1.3662	1.1600	0.9298	1.6600	0.9017
0.1700	5.4512	0.6700	1.3482	1.1700	0.9267	1.6700	0.9033
0.1800	5.1318	0.6800	1.3309	1.1800	0.9237	1.6800	0.9050
0.1900	4.8468	0.6900	1.3142	1.1900	0.9209	1.6900	0.9068
0.2000	4.5908	0.7000	1.2981	1.2000	0.9182	1.7000	0.9086
0.2100	4.3599	0.7100	1.2825	1.2100	0.9156	1.7100	0.9106
0.2200	4.1505	0.7200	1.2675	1.2200	0.9131	1.7200	0.9126
0.2300	3.9598	0.7300	1.2530	1.2300	0.9108	1.7300	0.9147
0.2400	3.7855	0.7400	1.2390	1.2400	0.9085	1.7400	0.9168
0.2500	3.6256	0.7500	1.2254	1.2500	0.9064	1.7500	0.9191
0.2600	3.4785	0.7600	1.2123	1.2600	0.9044	1.7600	0.9214
0.2700	3.3426	0.7700	1.1997	1.2700	0.9025	1.7700	0.9238
0.2800	3.2169	0.7800	1.1875	1.2800	0.9007	1.7800	0.9262
0.2900	3.1001	0.7900	1.1757	1.2900	0.8990	1.7900	0.9288
0.3000	2.9916	0.8000	1.1642	1.3000	0.8975	1.8000	0.9314
0.3100	2.8903	0.8100	1.1532	1.3100	0.8960	1.8100	0.9341
0.3200	2.7958	0.8200	1.1425	1.3200	0.8946	1.8200	0.9368
0.3300	2.7072	0.8300	1.1322	1.3300	0.8934	1.8300	0.9397
0.3400	2.6242	0.8400	1.1222	1.3400	0.8922	1.8400	0.9426
0.3500	2.5461	0.8500	1.1125	1.3500	0.8912	1.8500	0.9456
0.3600	2.4727	0.8600	1.1031	1.3600	0.8902	1.8600	0.9487
0.3700	2.4036	0.8700	1.0941	1.3700	0.8893	1.8700	0.9518
0.3800	2.3383	0.8800	1.0853	1.3800	0.8885	1.8800	0.9551
0.3900	2.2765	0.8900	1.0768	1.3900	0.8879	1.8900	0.9584
0.4000	2.2182	0.9000	1.0686	1.4000	0.8873	1.9000	0.9618
0.4100	2.1628	0.9100	1.0607	1.4100	0.8868	1.9100	0.9652
0.4200	2.1104	0.9200	1.0530	1.4200	0.8864	1.9200	0.9688
0.4300	2.0605	0.9300	1.0456	1.4300	0.8860	1.9300	0.9724
0.4400	2.0132	0.9400	1.0384	1.4400	0.8858	1.9400	0.9761
0.4500	1.9681	0.9500	1.0315	1.4500	0.8857	1.9500	0.9799
0.4600	1.9252	0.9600	1.0247	1.4600	0.8856	1.9600	0.9837
0.4700	1.8843	0.9700	1.0182	1.4700	0.8856	1.9700	0.9877
0.4800	1.8453	0.9800	1.0119	1.4800	0.8857	1.9800	0.9917
0.4900	1.8080	0.9900	1.0059	1.4900	0.8859	1.9900	0.9958
0.5000	1.7725	1.0000	1.0000	1.5000	0.8862	2.0000	1.0000

Weibull Distribution

- The random variable X with pdf

$$f(x) = \frac{\beta}{\delta}\left(\frac{x}{\delta}\right)^{\beta-1}\exp\left[-\left(\frac{x}{\delta}\right)^{\beta}\right], \quad \text{for } x > 0$$

is a **Weibull random variable** with scale parameter $\delta > 0$ and shape parameter $\beta > 0$

- The CDF for the Weibull distribution is

$$F(x) = 1 - \exp\left[-\left(\frac{x}{\delta}\right)^{\beta}\right]$$

- The mean of the Weibull distribution is

$$\mu = E(X) = \delta\Gamma\left(1 + \frac{1}{\beta}\right)$$

Weibull Problem

45. Suppose that fracture strength (MPa) of silicon nitride braze joints under certain conditions has a Weibull distribution with $\beta = 5$ and $\delta = 125$ (suggested by data in the article "Heat-Resistant Active Brazing of Silicon Nitride: Mechanical Evaluation of Braze Joints," (*Welding J.*, August 1997: 300s–304s).

a. What proportion of such joints have a fracture strength of at most 100? Between 100 and 150?

b. What strength value separates the weakest 50% of all joints from the strongest 50%?

c. What strength value characterizes the weakest 5% of all joints?

Figure 11: image

Weibull Practice Problems

Cdf of uniform

Tuesday, February 25, 2025 12:03 PM

$$f(x) = \frac{1}{b-a} \text{ for } a < x < b$$

$$F(x) = \begin{cases} \emptyset & \text{for } a > x \\ \int_a^x f(u)\,du \rightarrow \frac{x-a}{b-a} & \text{for } a < x < b \\ 1 & \text{for } x > b \end{cases}$$

$$\int_a^x f(u)\,du = \int_a^x \frac{1}{b-a}\,du = \left(\frac{1}{b-a}\right) u \Big|_{u=x}^{a}$$

Part f

Thursday, February 27, 2025 11:38 AM

find x s.t. $P(Z < x) = .55$

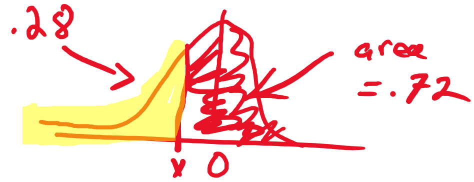
for a & b, we knew z and found $F(z) = \Phi(z)$

for part f we know $\Phi(z) = .55$ and want to find z

so $z = .13$

Part g

Thursday, February 27, 2025 11:43 AM

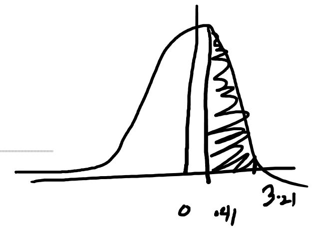


Find x s.t. $P(Z > x) = .72$

$x = -.58$

Standard normal pp

Thursday, February 27, 2025 11:48 AM



QUESTION 1

Let Z be a standard normal random variable, find P(0.41<Z<3.21).

- -0.364473
- 0.659761
- 0.340239
- 0.659097
- The correct answer is not provided.
- 0.000664

$$P(+.41 < Z < 3.21) = F(3.21) - F(.41)$$

$$= \Phi(3.21) - \Phi(.41)$$

$$= .999336 - .659097$$

$$= .340239$$

Screen clipping taken: 2/27/2025 11:48 AM

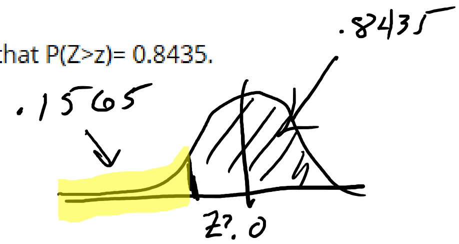
Standard normal pp

Thursday, February 27, 2025 11:53 AM

QUESTION 3

Let Z be a standard normal random variable, find the value z such that P(Z>z)= 0.8435.

- ◯ -1.01
- ◯ 2.64
- ◯ 0.95
- ◯ 1.01
- ◯ -0.95
- ◯ The correct answer is not provided.



$z = -1.01$

Screen clipping taken: 2/27/2025 11:54 AM

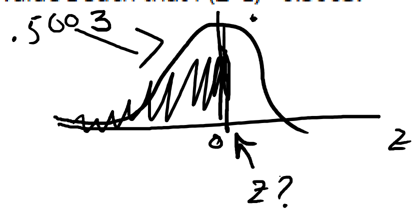
Standard normal pp

Thursday, February 27, 2025 11:58 AM

QUESTION 5

Let Z be a standard normal random variable, find the value z such that P(Z<z)= 0.5003.

- ○ -0.0
- ○ 3.72
- ○ -3.23
- ○ -3.72
- ○ -3.4
- ○ The correct answer is not provided.



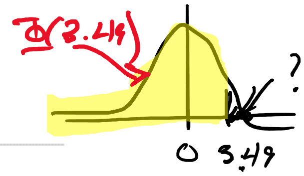
$Z = 0.0$

Screen clipping taken: 2/27/2025 11:59 AM

Standard normal pp

Thursday, February 27, 2025 12:01 PM

Attendance 1-D



QUESTION 7

Let Z be a standard normal random variable, find P(Z>3.49).

- 0.068272
- The correct answer is not provided.
- 0.999096
- 0.000242
- 0.000904
- 0.230874
- 0.999758

$P(Z > 3.49)$

$= 1 - \Phi(3.49)$

$= 1 - .999758$

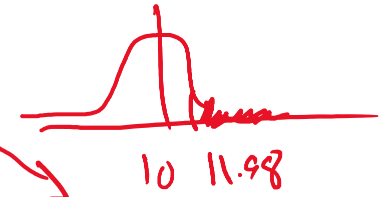
$= \underline{.000242}$

Screen clipping taken: 2/27/2025 12:11 PM

Part b

Tuesday, March 4, 2025 11:09 AM

$\mu = 10, \sigma = 2$



$P(X > 11.98)$

$$P(X > 11.98) = P\left(Z > \frac{11.98 - 10}{2}\right)$$

$$= P(Z > .99)$$

$$= 1 - \Phi(.99)$$

$$= 1 - .838913$$

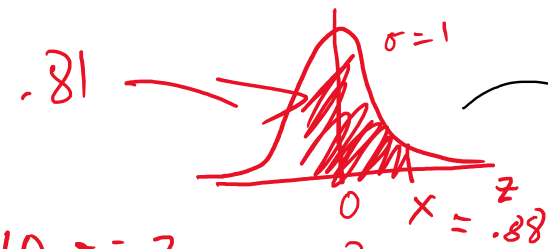
$$= \underline{.161087}$$

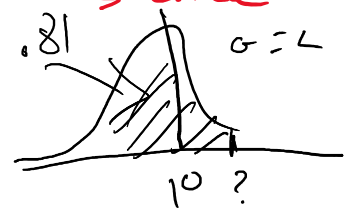
5.1.3 part f

Tuesday, March 4, 2025 11:14 AM

find x s.t. $P(X < x) = .81$

= area





Know $\mu = 10$, $\sigma = 2$, $z = .88$

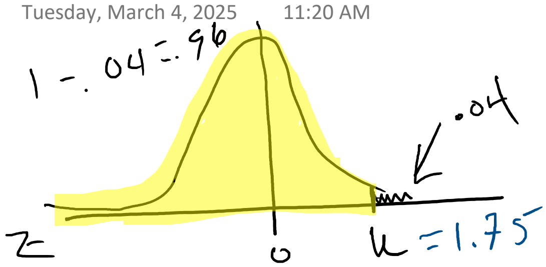
$$z = \frac{x - \mu}{\sigma}$$

$$.88 = \frac{x - 10}{2}$$

$$x = 10 + 2(.88)$$
$$= 11.76$$

5.1.3 part g

Tuesday, March 4, 2025 11:20 AM



Find u such that $P(X > u) = .04$

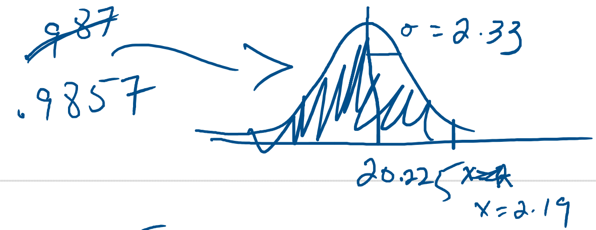
$$Z = \frac{X - \mu}{\sigma} \rightarrow 1.75 = \frac{X - 10}{2}$$

$$X = 10 + 2(1.75)$$

$$= \underline{13.50}$$

Normal practice problem

Tuesday, March 4, 2025 11:25 AM



QUESTION 1

μ σ

Let X be a normal random variable with mean (mu) = 20.225 and standard deviation (sigma) = 2.33, find the value x such that P(X<x)= 0.9857.

- ◯ 15.122
- ◯ 2.19
- ◯ 29.545
- ◯ -2.19
- ◯ 25.328
- ◯ The correct answer is not provided.

$$Z = \frac{X - \mu}{\sigma}$$

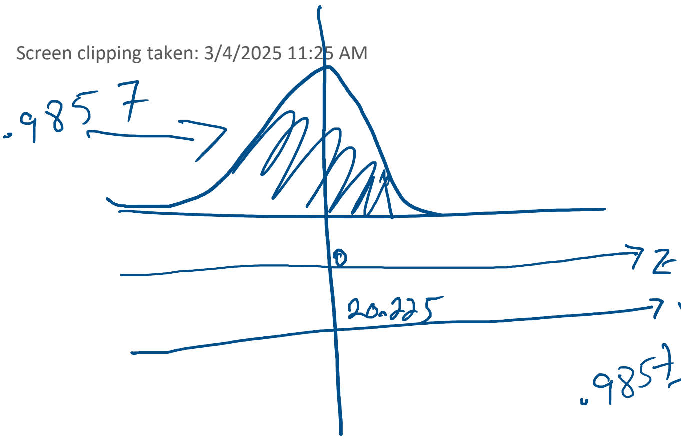
$$2.19 = \frac{X - 20.225}{2.33} \checkmark$$

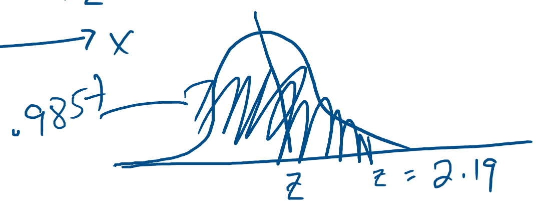
$$X = 2.19(2.33) + 20.225$$

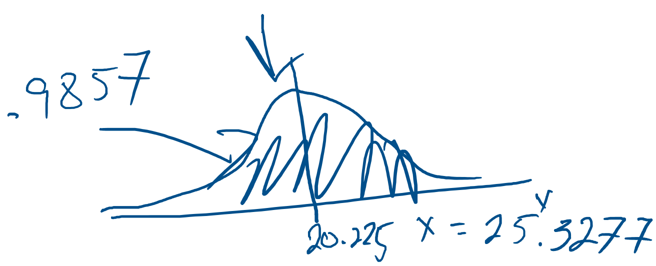
$$= Z\sigma + \mu$$

$$= 25.3277$$

Screen clipping taken: 3/4/2025 11:25 AM







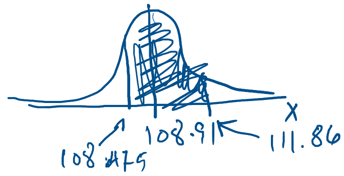
Normal pp

Tuesday, March 4, 2025 11:35 AM

QUESTION 3

Let X be a normal random variable with mean (mu) = 108.91 and standard deviation (sigma) = 3.626, find P(108.475<X<111.86).

- 0.661212
- -0.108711
- ~~The correct answer is not provided.~~
- 0.79103
- 0.338788
- 0.452242



Screen clipping taken: 3/4/2025 11:35 AM

$$P(108.475 < x < 111.86) = \Phi\left(\frac{111.86 - 108.91}{3.626}\right) - \Phi\left(\frac{108.475 - 108.91}{3.626}\right)$$

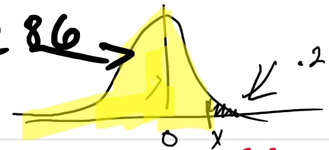
$$= \Phi(0.81) - \Phi(-.12)$$

$$= .791030 - .492242$$

$$= .298788$$

Tuesday, March 4, 2025 11:44 AM

.786



.79 ← z-score with $P(Z<z) \doteq .786$

QUESTION 5

Let X be a normal random variable with mean (mu) = 42.657 and standard deviation (sigma) = 1.964, find the value x such that P(X>x)= 0.214.

- The correct answer is not provided.
- 0.79 ✓
- 44.684
- 44.209
- -0.79 ✓
- 41.105

$$z = \frac{x-\mu}{\sigma} \rightarrow .79 = \frac{x-42.657}{1.964}$$

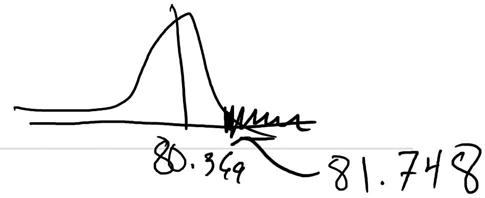
$$x = .79(1.964) + 42.657$$

$$= 44.20856$$

Screen clipping taken: 3/4/2025 11:44 AM

Normal pp

Tuesday, March 4, 2025 11:50 AM



QUESTION 7

Let X be a normal random variable with mean (mu) = 80.369 and standard deviation (sigma)=3.378, find P(X>81.748).

- 0.366782
- 0.659097
- The correct answer is not provided.
- 0.5
- 0.121137
- 0.340903

Screen clipping taken: 3/4/2025 11:50 AM

$$P(X>81.748) = 1 - \Phi\left(\frac{81.748-80.369}{3.378}\right)$$

$$= 1 - \Phi(.408)$$

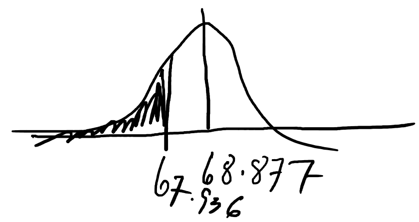
$$.41$$

$$= 1 - .659097$$

$$= .340903$$

Normal pp

Tuesday, March 4, 2025 11:54 AM



QUESTION 9

Let X be a normal random variable with mean (mu) = 68.877 and standard deviation (sigma)=1.976, find P(X<67.936).

- 0.315614
- 0.684386
- 0.5
- 0.899264
- 0.355533
- The correct answer is not provided.

$$P(X<67.936) = \Phi\left(\frac{X-\mu}{\sigma}\right)$$

$$= \Phi\left(\frac{67.936-68.877}{1.976}\right)$$

$$= \Phi(-.476)$$

$-.48$

$= .315614$

Screen clipping taken: 3/4/2025 11:55 AM

Attendance

Tuesday, March 4, 2025 12:00 PM

Line 1 - E

Line 2 - A) March 25 - Tuesday

B) March 27 - Thursday

Std Normal CDF

Thursday, March 6, 2025 11:08 AM

if $x < y$, then $F(x) < F(y)$

Mistake in Std. Normal CDF

errata → list of errors

Exponential pp

Thursday, March 6, 2025 11:43 AM

QUESTION 1

Let X be an exponential random variable with lambda=3.189, find P(X<0.3673).

- 0.0115
- 0.69
- 0.9885
- 0.31

Screen clipping taken: 3/6/2025 11:44 AM

$P(X < .3673) = F(.3673)$

$= 1 - e^{-\lambda x}$

$= 1 - e^{-3.189(.3673)}$

$= \cancel{1-e}\ \underline{.69004}$

Exponential pp

Thursday, March 6, 2025 11:47 AM

$F(x) = 1 - e^{-\lambda x}$

QUESTION 3

Let X be an exponential random variable with lambda=3.815, find P(0.2607 <X< 0.5486).

$= F(.5486) - F(.2607)$

$= \left[1 - e^{-3.815(.5486)}\right]$

$- \left[1 - e^{-3.815(.2607)}\right]$

- 0.3699
- 0.2466
- ~~-0.9406~~
- 0.8767

Screen clipping taken: 3/6/2025 11:47 AM

Exponential pp

Thursday, March 6, 2025 11:49 AM

QUESTION 7

Let X be an exponential random variable with lambda=3.924, find the value x such that P(X > x)= 0.872.

- 0.9673
- 0.3833
- 0.0349
- 0.5239
- 0.0327

Screen clipping taken: 3/6/2025 11:50 AM

$1 - F(x) = .872$

$1 - (1 - e^{-3.924x}) = .872$

$e^{-3.924x} = .872$

$\ln(e^{-3.924x}) = \ln .872$

$-3.924x = \ln .872$

$x = \frac{-\ln .872}{3.924}$

$= .0349$

Exponential pp

Thursday, March 6, 2025 11:54 AM

QUESTION 9

Let X be an exponential random variable with lambda=12.513. Find the value x such that P(X< x)= 0.4.

- 0.0732
- 0.0067
- 0.0408
- 0.9933
- 0.0026

Screen clipping taken: 3/6/2025 11:54 AM

$F(x) = .4$

$1 - e^{-12.513x} = .4$

$+e^{-12.513x} = +.6$

$\ln(e^{-12.513x}) = \ln .6$

$-12.513x = \ln .6$

$x = \frac{-\ln .6}{12.513}$

$= .0408$