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Wednesday, April 2, 2025 8:04 AM

Section 1

MANE 3332.04

Lecture 18, April 2

Agenda

- Midterm exam not graded
- Chapter 5 Material
- Chapter 6, time permitting
- Linear Combinations Practice Problems - due April 7, 2025
- Technical Report One - due April 7, 2025
- Attendance
- Questions?

Handouts

- Chapter 5 Slides
- Chapter 5 Slides Marked

Class Schedule

Monday Lecture	Wednesday Lecture
3/31: Chapter 6	4/2: Chapter 5
4/7: Chapter 7 & 8	4/9: Chapter 8, Case 1
4/14: Chapter 8: Case 2	4/16: Chapter 8: Case 3
4/21: Chapter 9, case 1	4/23: Chapter 9, Case 2
4/28: Chpater 9, Case 3	4/30: Chapter 11
5/5: Chapter 11	5/7: Review

11 classroom sessions plus Final Exam
Final Exam: Monday May 12, 2025, 8:00 - 9:45 am

Chapter Five

Statistics Theory Heavy

Bivariate X_1 & X_2 $f(x_1, x_2)$

- Joint Probability Distributions
- Contains eight sections
- We will only examine 5.4 (Covariance and Correlation) and 5.6 (linear functions of random variables), Central Limit Theorem

Covariance and Correlation

Covariance

- When two or more variables are defined on a probability space, it is useful to describe how they vary together
- A common measure of the relationship between two random variables is the **covariance**

Population Greek character

$$\sigma_{XY} = E(XY) - \mu_X\mu_Y$$

Covariance, continued

- Theoretically for two continuous random variables with joint probability distribution function $f_{XY}(x, y)$, the covariance is found by

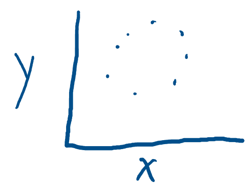
$$\sigma_{XY} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{XY}(x, y)dxdy - \mu_X\mu_Y$$

Discrete

$$\sigma_{xy} = \sum_{\text{all } x}\sum_{\text{all } y} xyf(x,y) - \mu_x \mu_y$$

circle

Covariance and Independence



- If X and Y are independent random variables,

$$\sigma_{XY} = 0$$

- However, $\sigma_{XY} = 0$ does not imply that X and Y are independent. Textbook mentions Figure 5-13(d)
- *SPECIAL CASE.* IF X and Y are normal random variables and have $\sigma_{XY} = 0$, then X and Y are independent

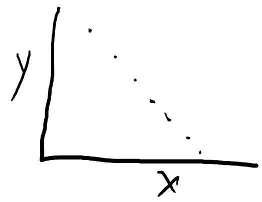
Sample Covariance

- To calculate the sample covariance use

$$\hat{\sigma}_{xy} = \qquad s_{XY} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$$

- Easily done in software

$\rho = 1$



$\rho = -1$

Correlation

- The correlation between two random variables X and Y is

ρ - rho

$$\rho_{XY} = \frac{\text{cov}(X, Y)}{\sqrt{V(X)V(Y)}} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

- For any two random variables X and Y

$$-1 \leq \rho_{XY} \leq 1$$

- If X and Y are independent $\rho_{XY} = 0$. The converse is not true.

Change $\sigma_{xy} \to S_{xy}$

$V(x) \to S_x^2$

$V(y) \to S_y^2$

Sample Correlation Coefficient

- To calculate the sample correlation coefficient,

$$r_{XY} = \frac{s_{XY}}{\sqrt{S_X^2 S_Y^2}}$$

Summary

- Correlation is a linear measure and will not work for non-linear relationships
- Correlation is a measure of association; it does not prove cause and effect relationships

  -Examine examples at Spurious Correlations website

Linear Functions of Random Variables

Functions of Random Variables

- Additive System. Let X be a random variable with mean μ and variance σ^2. Define a new random variable Y

$$Y = X + c$$

It follows that $E(X+c) = E(X) + \cancel{E(c)}$ c

$$E(Y) = E(X) + c = \mu + c$$
$$V(Y) = \boxed{V(X) + 0 = \sigma^2}$$

$V(Y) = V(X+c) = V(X) + \cancel{V(c)}$ 0.0

Linear Functions of Random Variables

Functions of Random Variables

- Multiplicative System. Consider the new random variable Y

$$Y = cX$$

It follows that

$$E(Y) = E(cX) = cE(X) = c\mu$$
$$V(Y) = V(cX) = c^2V(x) = c^2\sigma^2$$

important

$V(y) = E(x-\mu)^2$

~~Assum~~

Linear Combination

- A **linear combination** of the random variables $X_1, X_2, \ldots, X_n$ is

$$Y = c_1X_1 + c_2X_2 + \cdots + c_nX_n + C_0$$

- The mean of a linear combination of random variables is

$$E(Y) = c_1\mu_1 + c_2\mu_2 + \cdots + c_n\mu_n$$

independent

- The variance of a linear combination of random variables is

$$V(Y) = c_1^2\sigma_1^2 + c_2^2\sigma_2^2 + \cdots + c_n^2\sigma_n^2$$

Linear Combination of Non-independent R.V.

Let $X_1, X_2, \ldots, X_n$ be random variables with means $E(X_i) = \mu_i$, variances $V(X_i) = \sigma_i^2$ and covariances $\mathrm{Cov}(X_i, X_j)$ for $i, j = 1, 2, \ldots, n$ with $i < j$

- The linear combination is defined to be

$$Y = c_1X_1 + c_2X_2 + \cdots + c_nX_n$$

~~Same as~~

- The mean of Y is

$$E(Y) = c_1\mu_1 + c_2\mu_2 + \cdots + c_n\mu_n$$

Same as independent case

Linear Combination of Non-independent R.V.

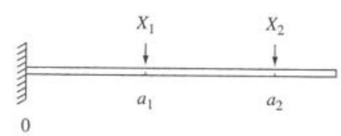
- and the variance is

$$V(Y) = c_1^2\sigma_1^2 + c_2^2\sigma_2^2 + \cdots + c_n^2\sigma_n^2 + 2\sum\sum_{i<j} c_i c_j \mathrm{Cov}(X_i, X_j)$$

Linear Combination Problem

Source: Devore (2000) Prob & Statistics

66. If two loads are applied to a cantilever beam as shown in the accompanying drawing, the bending moment at 0 due to the loads is $a_1X_1 + a_2X_2$.



a. Suppose that X_1 and X_2 are independent rv's with means 2 and 4 kips, respectively, and standard deviations .5 and 1.0 kip, respectively. If $a_1 = 5$ ft and $a_2 = 10$ ft, what is the expected bending moment and what is the standard deviation of the bending moment?

b. If X_1 and X_2 are normally distributed, what is the probability that the bending moment will exceed 75 kip-ft?

c. Suppose the positions of the two loads are random variables. Denoting them by A_1 and A_2, assume that these variables have means of 5 and 10 ft, respectively, that each has a standard deviation of .5, and that all A_i's and X_i's are independent of one another. What is the expected moment now?

d. For the situation of part (c), what is the variance of the bending moment?

e. If the situation is as described in part (a) except that $\mathrm{Corr}(X_1, X_2) = .5$ (so that the two loads are

Linear Combination Practice Problems

Central Limit Theorem

$\to \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

If $X_1, X_2, \ldots, X_n$ is a random sample of size n taken from a population with mean μ and variance σ^2, and if $\overline{X}$ is the sample mean, the limiting form of the distribution of

$$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}}$$

as $n \to \infty$, is the standard normal distribution

Central Limit Theorem

Concern for engineers is how big n have to be

- Incredibly useful theorem
- See example below
- n often does not have to be very large
  - If the population is continuous, unimodal and symmetric, often n can be as small as 4 or 5
  - Larger samples will be required in other situations
  - If $n \geq 30$ the normal approximation will work satisfactorily regardless of the shape of the population

CLT Illustration

Worst case

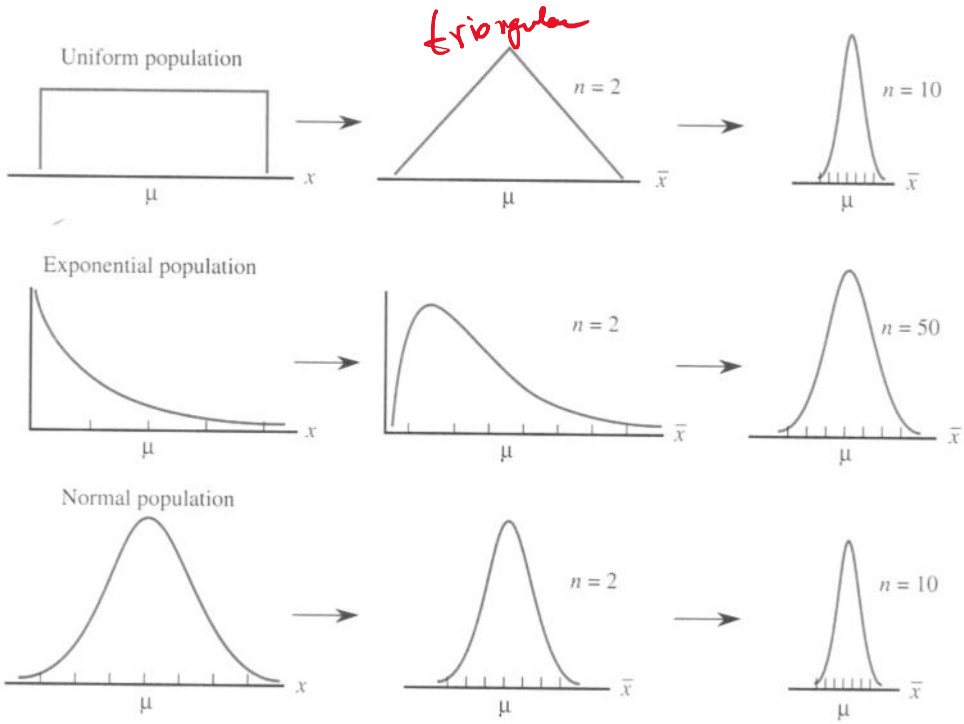


Figure 5.18 The Central Limit Theorem: The sampling distribution of $\bar{x}$ approaches

$a_1 = 5$ ft $\mu_{X_1} = 2$ $\sigma_{X_1} = 1/2$

$a_2 = 10$ ft $\mu_{X_2} = 4$ $\sigma_{X_2} = 1$

$$Y = 5X_1 + 10X_2$$

$c_1 = a_1 = 5$

$c_2 = a_2 = 10$

$$E(Y) = c_1 E(X_1) + c_2 E(X_2)$$
$$= 5(2) + 10(4) = 50$$

find σ_Y → always work variance & then take $\sqrt{}$

$$V(Y) = a_1^2 V(X_1) + a_2^2 V(X_2)$$
$$= 5^2\left(\tfrac{1}{2}\right)^2 + 10^2(1)^2 = 106.25$$
$$\sigma_Y = \sqrt{106.25} = \underline{10.3078}$$

$a_1 = 5$ $\quad \sigma_{x_1} = 1/2$ $\quad Corr(X_1, X_2) = 1/2$

$a_2 = 10$ $\quad \sigma_{x_2} = 1$

$Y = a_1 X_1 + a_2 X_2$

$$V(Y) = a_1^2 V(X_1) + a_2^2 V(X_2) + 2 \sum_{i<j}\sum a_i a_j Cov(X_1, X_2)$$

$$Corr(X_1, X_2) = \frac{1}{2} = \frac{Cov(X_1, X_2)}{\sqrt{V(X_1) V(X_2)}}$$

$$\frac{1}{2} = \frac{Cov(X_1, X_2)}{\sqrt{V(X_1) V(X_2)}} = \frac{Cov(X_1, X_2)}{\sqrt{(\frac{1}{2})^2 (1)^2}} \Rightarrow Cov(\;) = \frac{1}{2}\sqrt{(\frac{1}{2})^2 (1)^2}$$

$$= \frac{1}{4} = .25$$

$$V(Y) = 5^2 \left(\frac{1}{2}\right)^2 + 10^2 (1)^2 + 2(5)(10) .25$$

$$= 131.25$$

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$\boxed{\text{wrong: } -(2.743)^2 V(X_2)}$

$= -2.743$

QUESTION 5

Consider the linear combination Y = -2.588+(3.473)*X1+(-2.743)*X2. The random variables X1 and X2 are independent with E(X1)=28.793, E(X2)=32.829, V(X1)=17.599, and V(X2)=13.36. What is the value of V(Y)?

- 111.7531
- 4.5436
- ~~260.1503~~
- 312.7957
- 7.3601
- The correct answer is not provided.

$V(\cancel{-2.588} + 3.473X_1 - 2.743X_2) =$

$= (3.473)^2 V(X_1) + (-2.743)^2 V(X_2)$

$= (3.473)^2 (17.599) + (-2.743)^2 (13.36)$

$= \underline{312.7956}$

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QUESTION 7

$+2\sum\sum c$

Consider the linear combination Y = ~~-9.406~~+(6.195)*X1+(-1.95)*X2. The random variables X1 and X2 are not independent but have COV(X1,X2)=-10.518 with E(X1)=32.95, E(X2)=37.134, V(X1)=12.821, and V(X2)=18.89. What is the value of V(Y)?

- 309.7537
- 563.8739
- 817.994
- The correct answer is not provided.
- 122.248
- 420.2154

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$$V(Y) = c_1^2 V(X_1) + c_2^2 V(X_2) + 2\sum_{i<j}\sum c_i c_j Cov(X_1, X_2)$$

$$= (6.195)^2(12.821) + (-1.95)^2 18.89 + 2 c_1 c_2 Cov(X_1, X_2)$$

$$= (6.195)^2(12.821) + (-1.95)^2(18.89) + 2(6.195)(-1.95)(-10.518)$$

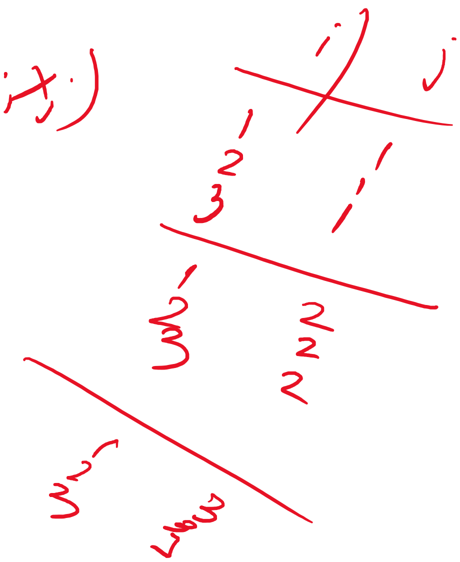
Suppose $n=3$ X_1, X_2, X_3

$Cov(\sim) \neq 0$

$\cancel{\#}$ $Y = c_1 X_1 + c_2 X_2 + c_3 X_3$

$$V(Y) = c_1^2 V(X_1) + c_2^2 V(X_2) + c_3^2 V(X_3) + 2 \sum_{i<j}\sum c_i c_j Cov(X_i, X_j)$$

think about $\sum\sum c_i c_j Cov(X_i, X_j)$



$i = j \to 3$ times

$Cov(X_1, X_1) = V(X_1)$

$Cov(X_2, X_2) = V(X_2)$

$Cov(X_3, X_3) = V(X_3)$

$Cov(X_1, X_2)$ & $Cov(X_2, X_1)$

$$2 \sum_{i<j}\sum c_i c_j (Cov(X_i, X_j)) = 2\{(c_1 c_2) Cov(X_1, X_2) + c_1 c_3 Cov(X_1, X_3) + c_2 c_3 Cov(X_2, X_3)\}$$