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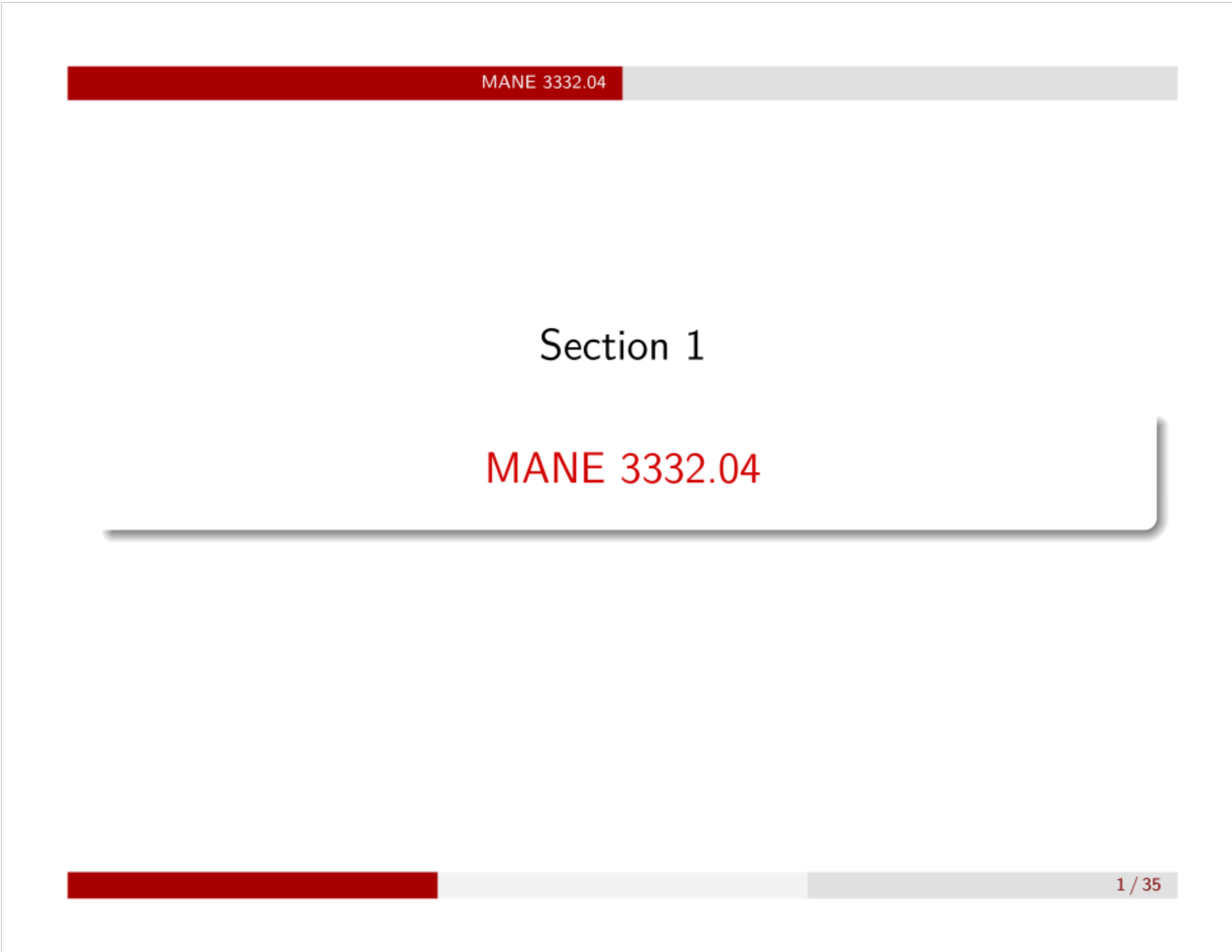


Lecture 19, April 7

- Topics:
  - Chapter 5: CLT
  - Chapter 6: Multivariate Statistical Analysis
  - Chapter 7: Definitions
  - Chapter 8: Interval Estimation
- Assignments:
  - Technical Report One due today
  - Linear Combination Practice Problems due today
  - Linear Combination Quiz (assigned 4/7/25, due 4/9/25)
- Attendance
- Return and discuss Test One
- Questions?

Handouts

- Chapter 5
  - Chapter 5 Slides
  - Chapter 5 Slides marked
- Chapter 6
  - Chapter 6 Slides
  - Chapter 6 Slides marked
- Chapter 7
  - Chapter 7 slides
  - Chapter
- Chapter 8
  - Chapter 8 slides
  - Chapter 8 slides marked
- Final Exam Handouts

Class Schedule

Monday Lecture	Wednesday Lecture
4/7: Chapter 7 & 8	4/9: Chapter 8, Case 1
4/14: Chapter 8: Case 2	4/16: Chapter 8: Case 3
4/21: Chapter 9, case 1	4/23: Chapter 9, Case 2
4/28: Chpater 9, Case 3	4/30: Chapter 11
5/5: Chapter 11	5/7: Review

10 Sessions plus final exam
Final Exam: Tuesday May 13, 2025 10:15 am - 12:00 pm

Chapter 8 Introduction

- Intervals are another method of performing estimation
- A confidence interval is an interval estimate on a population parameter (primary focus of this chapter)
- Three types of interval estimates
  - A confidence intervals bounds population or distribution parameters
  - A tolerance interval bounds a selected proportion of a distribution → 2 probabilities
  - A prediction interval bounds future observations from the population or distribution
- Interval estimates, especially confidence intervals are commonly used in science and engineering

μ or σ²

Tolerance: I want a 95% tolerance interval that contains 80% of the population

Chapter 9

Summary of One-Sample Hypothesis-Testing Procedures

Case	Null Hypothesis	Test Statistic	Alternative Hypothesis	Fixed Significance Level Criteria for Rejection	P-value	O.C. Curve Parameter	O.C. Curve Appendix Chart VII
1.	$H_0: \mu = \mu_0$, σ^2 known	$z_0 = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$	$H_1: \mu \neq \mu_0$	$\lvert z_0 \rvert > z_{\alpha/2}$	$P = 2[1 - \Phi(\lvert z_0 \rvert)]$	$d = \lvert \mu - \mu_0 \rvert / \sigma$	a, b
			$H_1: \mu > \mu_0$	$z_0 > z_\alpha$	Probability above z_0, $P = 1 - \Phi(z_0)$	$d = (\mu - \mu_0)/\sigma$	c, d
			$H_1: \mu < \mu_0$	$z_0 < -z_\alpha$	Probability below z_0, $P = \Phi(z_0)$	$d = (\mu_0 - \mu)/\sigma$	c, d
2.	$H_0: \mu = \mu_0$, σ^2 unknown	$t_0 = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$	$H_1: \mu \neq \mu_0$	$\lvert t_0 \rvert > t_{\alpha/2, n-1}$	Sum of the probability above $\lvert t_0 \rvert$ and below $-\lvert t_0 \rvert$	$d = \lvert \mu - \mu_0 \rvert / \sigma$	e, f
			$H_1: \mu > \mu_0$	$t_0 > t_{\alpha, n-1}$	Probability above t_0	$d = (\mu - \mu_0)/\sigma$	g, h
			$H_1: \mu < \mu_0$	$t_0 < -t_{\alpha, n-1}$	Probability below t_0	$d = (\mu_0 - \mu)/\sigma$	g, h
3.	$H_0: \sigma^2 = \sigma_0^2$	$\chi_0^2 = \frac{(n-1)s^2}{\sigma_0^2}$	$H_1: \sigma^2 \neq \sigma_0^2$	$\chi_0^2 > \chi^2_{\alpha/2, n-1}$ or $\chi_0^2 < \chi^2_{1-\alpha/2, n-1}$	See text Section 9.4.	$\lambda = \sigma/\sigma_0$	i, j
			$H_1: \sigma^2 > \sigma_0^2$	$\chi_0^2 > \chi^2_{\alpha, n-1}$		$\lambda = \sigma/\sigma_0$	k, l
			$H_1: \sigma^2 < \sigma_0^2$	$\chi_0^2 < \chi^2_{1-\alpha, n-1}$		$\lambda = \sigma/\sigma_0$	m, n
4.	$H_0: p = p_0$	$z_0 = \frac{x - np_0}{\sqrt{np_0(1-p_0)}}$	$H_1: p \neq p_0$	$\lvert z_0 \rvert > z_{\alpha/2}$	$p = 2[1 - \Phi(\lvert z_0 \rvert)]$	3-4	3-4
			$H_1: p > p_0$	$z_0 > z_\alpha$	Probability above z_0, $p = 1 - \Phi(z_0)$	3-4	3-4
			$H_1: p < p_0$	$z_0 < -z_\alpha$	Probability below z_0, $P = \Phi(z_0)$	3-4	3-4

Chapter 8

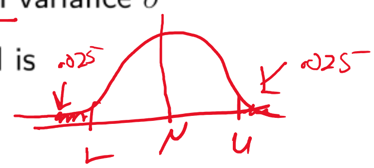
Summary of One-Sample Confidence Interval Procedures

Case	Problem Type	Point Estimate	Two-sided $100(1-\alpha)$ Percent Confidence Interval
1.	Mean μ, variance σ^2 known	$\bar{x}$	$\bar{x} - z_{\alpha/2}\sigma/\sqrt{n} \le \mu \le \bar{x} + z_{\alpha/2}\sigma/\sqrt{n}$
2.	Mean μ of a normal distribution, variance σ^2 unknown	$\bar{x}$	$\bar{x} - t_{\alpha/2, n-1}s/\sqrt{n} \le \mu \le \bar{x} + t_{\alpha/2, n-1}s/\sqrt{n}$
3.	Variance σ^2 of a normal distribution	s^2	$\frac{(n-1)s^2}{\chi^2_{\alpha/2, n-1}} \le \sigma^2 \le \frac{(n-1)s^2}{\chi^2_{1-\alpha/2, n-1}}$
4.	Proportion or parameter of a binomial distribution p	$\hat{p}$	$\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

Confidence Interval on the Mean of a normal distribution, variance known (Case 1)

→ in practice, not common

- Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from a normal population with unknown mean μ and known variance σ^2
- A general expression for a confidence interval is



if $\alpha = .05$

$$P\left[L \le \mu \le U\right] = 1 - \alpha$$

- Using the sample results we calculate a $100(1-\alpha)\%$ confidence of the form

$l = \bar{X} - Z_{\alpha/2}\, \sigma/\sqrt{n}$

$u = \bar{X} + Z_{\alpha/2}\, \sigma/\sqrt{n}$

b

$$l \le \mu \le u$$

- A $100(1-\alpha)\%$ confidence interval for the mean of a normal distribution with variance known is

Attendance 1-A

Problem 8-12,part a (6th edition)

8-12. The life in hours of a 75-watt light bulb is known to be normally distributed with $\sigma = 25$ hours. A random sample of 20 bulbs has a mean life of $\bar{x} = 1014$ hours.

(a) Construct a 95% two-sided confidence interval on the mean life.

(b) Construct a 95% lower-confidence bound on the mean life.

Figure 2: image

Interpreting Confidence Intervals

Montgomery gives the following statement regarding the correct interpretation of confidence intervals.

> *The correct interpretation lies in the realization that a CI is a random interval because in the probability statement defining the end-points of the interval, L and U are random variables. Consequently, the correct interpretation of a* $100(1-\alpha)\%$ *CI depends on the relative frequency view of probability. Specifically, if an infinite number of random samples are collected and a* $100(1-\alpha)\%$ *confidence interval for* μ *is computed from each sample,* $100(1-\alpha)\%$ *of these intervals will contain the true value of* μ.

One-sided Confidence Bounds

It is possible to construct on-sided confidence bounds

- A $100(1-\alpha)\%$ upper-confidence bound for μ is

$$\mu \le u = \bar{x} + z_\alpha \frac{\sigma}{\sqrt{n}}$$

- A $100(1-\alpha)\%$ lower-confidence bound for μ is

$$\bar{x} - z_\alpha \frac{\sigma}{\sqrt{n}} = l \le \mu$$

Sample Size Considerations

If $\bar{x}$ is used as an estimate of μ, we can be $100(1-\alpha)\%$ confident that the error $|\bar{x}-\mu|$ will not exceed a specified amount E when the sample size is

$$n = \left(\frac{z_{\alpha/2}\sigma}{E}\right)^2$$

Problem 8–12,part b (6th edition)

8-12. The life in hours of a 75-watt light bulb is known to be normally distributed with $\sigma = 25$ hours. A random sample of 20 bulbs has a mean life of $\bar{x} = 1014$ hours.

(a) Construct a 95% two-sided confidence interval on the mean life.

(b) Construct a 95% lower-confidence bound on the mean life.

Figure 3: image

A Large Sample CI for μ

- When n is large (say greater than or equal to 40), the central limit theorem can be used
- It states that $\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}$ is approximately a standard normal random variable.
- Thus, we can replace the quantity $\sigma/\sqrt{n}$ with $S/\sqrt{n}$ and still use the quantiles of the normal distribution to construct a confidence interval.

$$\bar{x} - z_{\alpha/2}\frac{s}{\sqrt{n}} \leq \mu \leq \bar{x} + z_{\alpha/2}\frac{s}{\sqrt{n}}$$

- What assumption did we relax and why?

Chapter 8, Case 1 Practice Problems

Confidence Interval for the mean of Normal distribution with variance unknown (Case 2)

The t distribution

- Definition.

Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal distribution with unknown mean μ and unknown variance σ^2. The random variable

$$T = \frac{\overline{X} - \mu}{S/\sqrt{n}}$$

has a t distribution with $n-1$ degrees of freedom.

- A table of percentage points (quantiles) is given in Appendix A Table 5
- Figure 8-4 on page 180 shows the relationship between the t and normal distributions.
- Figure 8-5 explains the percentage points of the t distribution

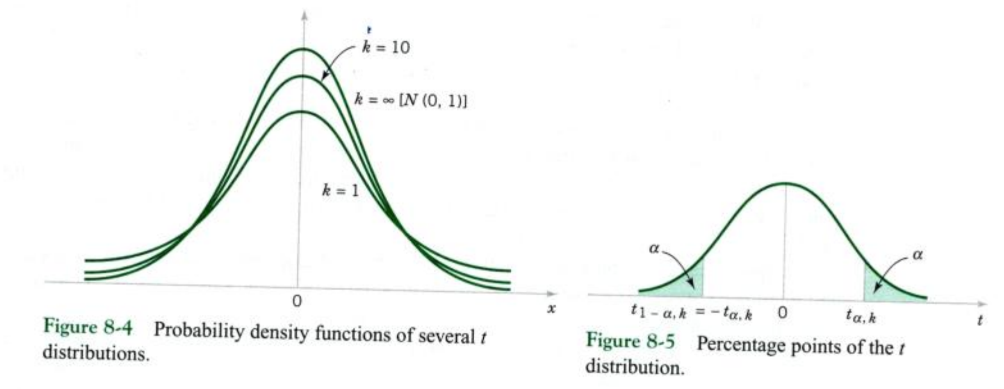


Figure 4: image

Confidence interval definition

- Using the t distribution it is possible to construct CIs

If $\bar{x}$ and s are the mean and standard deviation of a random sample from a normal distribution with unknown variance σ^2, a $100(1-\alpha)\%$ confidence interval on μ is given by

$$\bar{x} - t_{\alpha/2,n-1}\frac{s}{\sqrt{n}} \le \mu \le \bar{x} + t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}$$

where $t_{\alpha/2,n-1}$ is the upper $100(\alpha/2)$ percentage point of the t distribution with $n-1$ degrees of freedom.

Problem 8-30 (6th edition)

8-30. An article in *Nuclear Engineering International* (February 1988, p. 33) describes several characteristics of fuel rods used in a reactor owned by an electric utility in Norway. Measurements on the percentage of enrichment of 12 rods were reported as follows:

2.94	3.00	2.90	2.75	3.00	2.95
2.90	2.75	2.95	2.82	2.81	3.05

(a) Use a normal probability plot to check the normality assumption.

(b) Find a 99% two-sided confidence interval on the mean percentage of enrichment. Are you comfortable with the statement that the mean percentage of enrichment is 2.95 percent? Why?

Figure 5: image

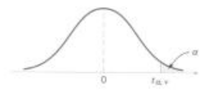


Table IV Percentage Points $t_{\alpha,\nu}$ of the t-Distribution

ν \ α	.40	.25	.10	.05	.025	.01	.005	.0025	.001	.0005
1	.325	1.000	3.078	6.314	12.706	31.821	63.657	127.32	318.31	636.62
2	.289	.816	1.886	2.920	4.303	6.965	9.925	14.089	23.326	31.598
3	.277	.765	1.638	2.353	3.182	4.541	5.841	7.453	10.213	12.924
4	.271	.741	1.533	2.132	2.776	3.747	4.604	5.598	7.173	8.610
5	.267	.727	1.476	2.015	2.571	3.365	4.032	4.773	5.893	6.869
6	.265	.718	1.440	1.943	2.447	3.143	3.707	4.317	5.208	5.959
7	.263	.711	1.415	1.895	2.365	2.998	3.499	4.029	4.785	5.408
8	.262	.706	1.397	1.860	2.306	2.896	3.355	3.833	4.501	5.041
9	.261	.703	1.383	1.833	2.262	2.821	3.250	3.690	4.297	4.781
10	.260	.700	1.372	1.812	2.228	2.764	3.169	3.581	4.144	4.587
11	.260	.697	1.363	1.796	2.201	2.718	3.106	3.497	4.025	4.437
12	.259	.695	1.356	1.782	2.179	2.681	3.055	3.428	3.930	4.318
13	.259	.694	1.350	1.771	2.160	2.650	3.012	3.372	3.852	4.221
14	.258	.692	1.345	1.761	2.145	2.624	2.977	3.326	3.787	4.140
15	.258	.691	1.341	1.753	2.131	2.602	2.947	3.286	3.733	4.073
16	.258	.690	1.337	1.746	2.120	2.583	2.921	3.252	3.686	4.015
17	.257	.689	1.333	1.740	2.110	2.567	2.898	3.222	3.646	3.965
18	.257	.688	1.330	1.734	2.101	2.552	2.878	3.197	3.610	3.922
19	.257	.688	1.328	1.729	2.093	2.539	2.861	3.174	3.579	3.883
20	.257	.687	1.325	1.725	2.086	2.528	2.845	3.153	3.552	3.850
21	.257	.686	1.323	1.721	2.080	2.518	2.831	3.135	3.527	3.819
22	.256	.686	1.321	1.717	2.074	2.508	2.819	3.119	3.505	3.792
23	.256	.685	1.319	1.714	2.069	2.500	2.807	3.104	3.485	3.767
24	.256	.685	1.318	1.711	2.064	2.492	2.797	3.091	3.467	3.745
25	.256	.684	1.316	1.708	2.060	2.485	2.787	3.078	3.450	3.725
26	.256	.684	1.315	1.706	2.056	2.479	2.779	3.067	3.435	3.707
27	.256	.684	1.314	1.703	2.052	2.473	2.771	3.057	3.421	3.690
28	.256	.683	1.313	1.701	2.048	2.467	2.763	3.047	3.408	3.674
29	.256	.683	1.311	1.699	2.045	2.462	2.756	3.038	3.396	3.659
30	.256	.683	1.310	1.697	2.042	2.457	2.750	3.030	3.385	3.646
40	.255	.681	1.303	1.684	2.021	2.423	2.704	2.971	3.307	3.551
60	.254	.679	1.296	1.671	2.000	2.390	2.660	2.915	3.232	3.460
120	.254	.677	1.289	1.658	1.980	2.358	2.617	2.860	3.160	3.373
∞	.253	.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

ν = degrees of freedom.

One-sided confidence bounds

- Are easy to construct
- Use only the appropriate upper or lower bound
- Change $t_{\alpha/2,n-1}$ to $t_{\alpha,n-1}$

Chapter 8, Case 2 Practice Problems

Confidence Interval for σ^2 and σ (Case 3)

- Section 8-3 presents a CI for σ^2 or σ
- Requires the χ^2 (chi-squared) distribution

Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal distribution with mean μ and variance σ^2 and let S^2 be the sample variance. Then the random variable

$$\chi^2 = \frac{(n-1)S^2}{\sigma^2}$$

has a chi-square (χ^2) distribution with $n-1$ degrees of freedom

- A table of the upper percentage points of the χ^2 distribution are given in Table 4 in the appendix
- Figure 8-9 on page 183 explains the percentage points of the χ^2 distribution

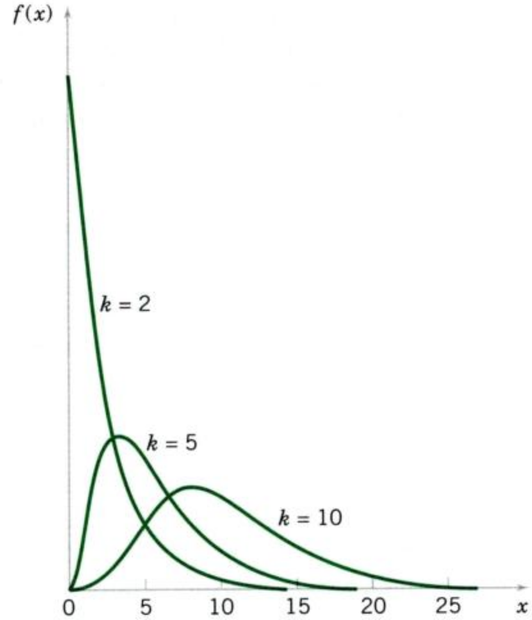


Figure 8-8 Probability density functions of several χ^2 distributions.

Figure 7: image

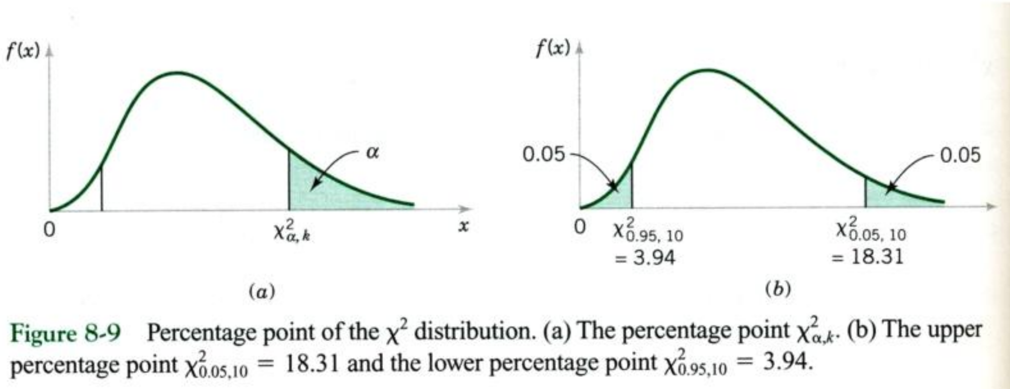


Figure 8: image

Confidence Intervals for σ^2 and σ

If s^2 is the sample variance from a random sample of n observations from a normal distribution with unknown variance σ^2, then a $100(1-\alpha)\%$ confidence interval on σ^2 is

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}} \le \sigma^2 \le \frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}}$$

where $\chi^2_{\alpha/2,n-1}$ and $\chi^2_{1-\alpha/2,n-1}$ are the upper and lower $100\alpha/2$ percentage points of the χ^2-distribution with $n-1$ degrees of freedom

Problem 8-36 (6th edition)

8-36. The sugar content of the syrup in canned peaches i normally distributed. A random sample of $n = 10$ cans yield a sample standard deviation of $s = 4.8$ milligrams. Find 95% two-sided confidence interval for σ.

Figure 9: image

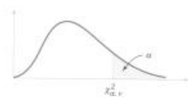


Table III Percentage Points $\chi^2_{\alpha,\nu}$ of the Chi-Squared Distribution

ν \ α	.995	.990	.975	.950	.900	.500	.100	.050	.025	.010	.005
1	.00+	.00+	.00+	.00+	.02	.45	2.71	3.84	5.02	6.63	7.88
2	.01	.02	.05	.10	.21	1.39	4.61	5.99	7.38	9.21	10.60
3	.07	.11	.22	.35	.58	2.37	6.25	7.81	9.35	11.34	12.84
4	.21	.30	.48	.71	1.06	3.36	7.78	9.49	11.14	13.28	14.86
5	.41	.55	.83	1.15	1.61	4.35	9.24	11.07	12.83	15.09	16.75
6	.68	.87	1.24	1.64	2.20	5.35	10.65	12.59	14.45	16.81	18.55
7	.99	1.24	1.69	2.17	2.83	6.35	12.02	14.07	16.01	18.48	20.28
8	1.34	1.65	2.18	2.73	3.49	7.34	13.36	15.51	17.53	20.09	21.96
9	1.73	2.09	2.70	3.33	4.17	8.34	14.68	16.92	19.02	21.67	23.59
10	2.16	2.56	3.25	3.94	4.87	9.34	15.99	18.31	20.48	23.21	25.19
11	2.60	3.05	3.82	4.57	5.58	10.34	17.28	19.68	21.92	24.72	26.76
12	3.07	3.57	4.40	5.23	6.30	11.34	18.55	21.03	23.34	26.22	28.30
13	3.57	4.11	5.01	5.89	7.04	12.34	19.81	22.36	24.74	27.69	29.82
14	4.07	4.66	5.63	6.57	7.79	13.34	21.06	23.68	26.12	29.14	31.32
15	4.60	5.23	6.27	7.26	8.55	14.34	22.31	25.00	27.49	30.58	32.80
16	5.14	5.81	6.91	7.96	9.31	15.34	23.54	26.30	28.85	32.00	34.27
17	5.70	6.41	7.56	8.67	10.09	16.34	24.77	27.59	30.19	33.41	35.72
18	6.26	7.01	8.23	9.39	10.87	17.34	25.99	28.87	31.53	34.81	37.16
19	6.84	7.63	8.91	10.12	11.65	18.34	27.20	30.14	32.85	36.19	38.58
20	7.43	8.26	9.59	10.85	12.44	19.34	28.41	31.41	34.17	37.57	40.00
21	8.03	8.90	10.28	11.59	13.24	20.34	29.62	32.67	35.48	38.93	41.40
22	8.64	9.54	10.98	12.34	14.04	21.34	30.81	33.92	36.78	40.29	42.80
23	9.26	10.20	11.69	13.09	14.85	22.34	32.01	35.17	38.08	41.64	44.18
24	9.89	10.86	12.40	13.85	15.66	23.34	33.20	36.42	39.36	42.98	45.56
25	10.52	11.52	13.12	14.61	16.47	24.34	34.28	37.65	40.65	44.31	46.93
26	11.16	12.20	13.84	15.38	17.29	25.34	35.56	38.89	41.92	45.64	48.29
27	11.81	12.88	14.57	16.15	18.11	26.34	36.74	40.11	43.19	46.96	49.65
28	12.46	13.57	15.31	16.93	18.94	27.34	37.92	41.34	44.46	48.28	50.99
29	13.12	14.26	16.05	17.71	19.77	28.34	39.09	42.56	45.72	49.59	52.34
30	13.79	14.95	16.79	18.49	20.60	29.34	40.26	43.77	46.98	50.89	53.67
40	20.71	22.16	24.43	26.51	29.05	39.34	51.81	55.76	59.34	63.69	66.77
50	27.99	29.71	32.36	34.76	37.69	49.33	63.17	67.50	71.42	76.15	79.49
60	35.53	37.48	40.48	43.19	46.46	59.33	74.40	79.08	83.30	88.38	91.95
70	43.28	45.44	48.76	51.74	55.33	69.33	85.53	90.53	95.02	100.42	104.22
80	51.17	53.54	57.15	60.39	64.28	79.33	96.58	101.88	106.63	112.33	116.32
90	59.20	61.75	65.65	69.13	73.29	89.33	107.57	113.14	118.14	124.12	128.30
100	67.33	70.06	74.22	77.93	82.36	99.33	118.50	124.34	129.56	135.81	140.17

ν = degrees of freedom.

One-sided confidence bounds

- Are easy to construct
- Use only the appropriate upper or lower bound
- Change $\chi^2_{\alpha/2,n-1}$ to $\chi^2_{\alpha,n-1}$ or $\chi^2_{1-\alpha/2,n-1}$ to $\chi^2_{1-\alpha,n-1}$
- See eqn (8-20) on page 184

Chapter 8, Case 3 Practice Problems

Large-Sample CI for a Population Proportion (Case 4)

- Recall from chapter 4, that the sampling distribution of $\widehat{P}$ is approximately normal with mean p and variance $p(1-p)/p$, if n is not too close to either 0 or 1 and if n is relatively large.
- Typically, we require both $np \geq 5$ and $n(1-p) \geq 5$

f n is large, the distribution of

$$Z = \frac{X - np}{\sqrt{np(1-p)}} = \frac{\widehat{P} - p}{\sqrt{\frac{p(1-p)}{n}}}$$

is approximately standard normal.

If $\hat{p}$ is the proportion of observations in a random sample of size n that belongs to a class of interest, an approximate $100(1-\alpha)\%$ confidence interval on the proportion p of the population that belongs to this class is

$$\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

where $z_{\alpha/2}$ is the upper $\alpha/2$ percentage point of the standard normal distribution

Other Considerations

- We can select a sample so that we are $100(1-\alpha)\%$ confident that error $E = |p - \widehat{P}|$ using

$$n = \left(\frac{z_{\alpha/2}}{E}\right)^2 p(1-p)$$

- An upper bound on is given by

$$n = \left(\frac{z_{\alpha/2}}{E}\right)^2 (0.25)$$

- One-sided confidence bounds are given in eqn (8-26) on page 187

Guidelines for Constructing Confidence Intervals

- Review excellent guide given in Table 8-1

Other Interval Estimates

- When we want to predict the value of a single value in the future, a **prediction interval** is used
- A **tolerance interval** captures $100(1-\alpha)\%$ of observations from a distribution

Prediction Interval for a Normal Distribution

- Excellent discussion on pages 189 - 190
- A $100(1-\alpha)\%$ PI on a single future observation from a normal distribution is given by

$$\bar{x} - t_{\alpha/2,n-1} s\sqrt{1+\frac{1}{n}} \le X_{n+1} \le \bar{x} + t_{\alpha/2,n-1} s\sqrt{1+\frac{1}{n}}$$

Tolerance Intervals for a Normal Distribution

- A **tolerance interval** to contain at least $\gamma\%$ of the values in a normal population with confidence level $100(1-\alpha)\%$ is

$$\bar{x} - ks, \bar{x} + ks$$

where k is a tolerance interval factor for the normal distribution found in appendix A Table XII. Values are given for $1-\alpha$=0.9, 0.95 and 0.99 confidence levels and for $\gamma = .90$, .95, and .99% probability of coverage

- One-sided tolerance bounds can also be computed. The factors are also in Table XII