

Section 1

MANE 3332.04

Lecture 18, April 2

Agenda

- Midterm exam not graded
- Chapter 5 Material
- Chapter 6, time permitting
- Linear Combinations Practice Problems - due April 7, 2025
- Technical Report One - due April 7, 2025
- Attendance
- Questions?

Handouts

- Chapter 5 Slides
- Chapter 5 Slides Marked

Class Schedule

Monday Lecture	Wednesday Lecture
3/31: Chapter 6	4/2: Chapter 5
4/7: Chapter 7 & 8	4/9: Chapter 8, Case 1
4/14: Chapter 8: Case 2	4/16: Chapter 8: Case 3
4/21: Chapter 9, case 1	4/23: Chapter 9, Case 2
4/28: Chapter 9, Case 3	4/30: Chapter 11
5/5: Chapter 11	5/7: Review

11 classroom sessions plus Final Exam

Final Exam: Monday May 12, 2025, 8:00 - 9:45 am

Chapter Five

- Joint Probability Distributions
- Contains eight sections
- We will only examine 5.4 (Covariance and Correlation) and 5.6 (linear functions of random variables)

Covariance and Correlation

Covariance

- When two or more variables are defined on a probability space, it is useful to describe how they vary together
- A common measure of the relationship between two random variables is the **covariance**

$$\sigma_{XY} = E(XY) - \mu_X \mu_Y$$

Covariance, continued

- Theoretically for two continuous random variables with joint probability distribution function $f_{XY}(x, y)$, the covariance is found by

$$\sigma_{XY} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf_{XY}(x, y)dx dy - \mu_X\mu_Y$$

Covariance and Independence

- If X and Y are independent random variables,

$$\sigma_{XY} = 0$$

- However, $\sigma_{XY} = 0$ does not imply that X and Y are independent. Textbook mentions Figure 5-13(d)
- *SPECIAL CASE.* IF X and Y are normal random variables and have $\sigma_{XY} = 0$, then X and Y are independent

Sample Covariance

- To calculate the sample covariance use

$$s_{XY} = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})$$

- Easily done in software

Correlation

- The correlation between two random variables X and Y is

$$\rho_{XY} = \frac{\text{cov}(X, Y)}{\sqrt{V(X)V(Y)}} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

- For any two random variables X and Y

$$-1 \leq \rho_{XY} \leq 1$$

- If X and Y are independent $\rho_{XY} = 0$. The converse is not true.

Sample Correlation Coefficient

- To calculate the sample correlation coefficient,

$$r_{XY} = \frac{s_{XY}}{\sqrt{S_X^2 S_Y^2}}$$

Summary

- Correlation is a linear measure and will not work for non-linear relationships
- Correlation is a measure of association; it does not prove cause and effect relationships
 - Examine examples at [Spurious Correlations](#) website

Linear Functions of Random Variables

Functions of Random Variables

- Additive System. Let X be a random variable with mean μ and variance σ^2 . Define a new random variable Y

$$Y = X + c$$

It follows that

$$\begin{aligned} E(Y) &= E(X) + c = \mu + c \\ V(Y) &= V(X) + 0 = \sigma^2 \end{aligned}$$

Linear Functions of Random Variables

Functions of Random Variables

- Multiplicative System. Consider the new random variable Y

$$Y = cX$$

It follows that

$$E(Y) = E(cX) = cE(X) = c\mu$$

$$V(Y) = V(cX) = c^2 V(X) = c^2 \sigma^2$$

Linear Combination

- A **linear combination** of the random variables X_1, X_2, \dots, X_n is

$$Y = c_1X_1 + c_2X_2 + \dots + c_nX_n$$

- The mean of a linear combination of random variables is

$$E(Y) = c_1\mu_1 + c_2\mu_2 + \dots + c_n\mu_n$$

- The variance of a linear combination of random variables is

$$V(Y) = c_1^2\sigma_1^2 + c_2^2\sigma_2^2 + \dots + c_n^2\sigma_n^2$$

Linear Combination of Non-independent R.V.

Let X_1, X_2, \dots, X_n be random variables with means $E(X_i) = \mu_i$, variances $V(X_i) = \sigma_i^2$ and covariances $\text{Cov}(X_i, X_j)$ for $i, j = 1, 2, \dots, n$ with $i < j$

- The linear combination is defined to be

$$Y = c_1X_1 + c_2X_2 + \dots + c_nX_n$$

- The mean of Y is

$$E(Y) = c_1\mu_1 + c_2\mu_2 + \dots + c_n\mu_n$$

Linear Combination of Non-independent R.V.

- and the variance is

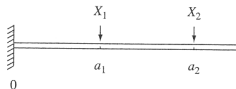
$$V(Y) = c_1^2 \sigma^2 + c_2^2 \sigma^2 + \cdots + c_n^2 \sigma_n^2 \\ + 2 \sum \sum_{i < j} c_i c_j \text{Cov}(X_i, X_j)$$

Linear Combination Problem

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Source: Devore (2000) Prob & Statistics

66. If two loads are applied to a cantilever beam as shown in the accompanying drawing, the bending moment at 0 due to the loads is $a_1X_1 + a_2X_2$.



- Suppose that X_1 and X_2 are independent rv's with means 2 and 4 kips, respectively, and standard deviations .5 and 1.0 kip, respectively. If $a_1 = 5$ ft and $a_2 = 10$ ft, what is the expected bending moment and what is the standard deviation of the bending moment?
- If X_1 and X_2 are normally distributed, what is the probability that the bending moment will exceed 75 kip-ft?
- Suppose the positions of the two loads are random variables. Denoting them by A_1 and A_2 , assume that these variables have means of 5 and 10 ft, respectively, that each has a standard deviation of .5, and that all A_i 's and X_i 's are independent of one another. What is the expected moment now?
- For the situation of part (c), what is the variance of the bending moment?
- If the situation is as described in part (a) except that $\text{Corr}(X_1, X_2) = .5$ (so that the two loads are

Linear Combination Practice Problems

Central Limit Theorem

If X_1, X_2, \dots, X_n is a random sample of size n taken from a population with mean μ and variance σ^2 , and if \bar{X} is the sample mean, the limiting form of the distribution of

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

as $n \rightarrow \infty$, is the standard normal distribution

Central Limit Theorem

- Incredibly useful theorem
- See example below
- n often does not have to be very large
 - If the population is continuous, unimodal and symmetric, often n can be as small as 4 or 5
 - Larger samples will be required in other situations
 - If $n \geq 30$ the normal approximation will work satisfactorily regardless of the shape of the population

CLT Illustration

5.6 Describing Sampling Distributions

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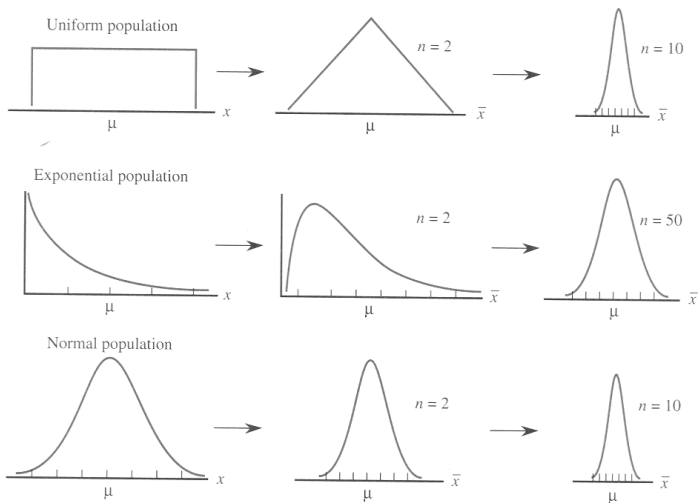


Figure 5.18 The Central Limit Theorem: The sampling distribution of \bar{x} approaches