Attendance : 1 - E

MANE 3332.05

LECTURE 13

Agenda

- Continue Chapter 4 lectures - Exponential and Weibull Distributions
- Normal Practice Problems (assigned 10/9/2025, due 10/14/2025)
- Normal Quiz (assigned 10/14/2025, due 10/16/2026)
- Exponential Practice Problems (assigned 10/14/2025, due 10/16/2025)
- Schedule

Handouts

- Lecture 13 Slides - Powerpoint
- Lecture 13 Slides - marked (pdf)

Tuesday Date and Topic(s)	Thursday Date and Topic(s)
10/14: Exponential and Weibull distributions	**10/16:** Chapter 5 (not on midterm)
10/21: Midterm Review	**10/23:** Midterm Exam

Exponential Distribution

- The exponential distribution is widely used in the area of reliability and life-test data.
- Ostle, et. al. (1996) list the following applications of the exponential distribution
  - the number of feet between two consecutive erroneous records on a computer tape,
  - the lifetime of a component of a particular device,
  - the length of a life of a radioactive material and
  - the time to the next customer service call at a service desk

Exponential Distribution

- The PDF for an exponential distribution with parameter $\lambda > 0$ is

$$f(x) = \lambda e^{-\lambda x}, \quad \text{for } 0 \leq x < \infty$$

- The mean of X is

$$\mu = E(X) = \frac{1}{\lambda} = \int_{-\infty}^{\infty} x f(x)\,dx$$

- The variance of X is

$$\sigma^2 = V(X) = \frac{1}{\lambda^2}$$

Note that other authors define $f(x) = \frac{1}{\theta} e^{-x/\theta}$. Either definition is acceptable. However one must be aware of which definition is being used.

The Exponential CDF

The CDF for the exponential distribution is easy to derive

$$
\begin{aligned}
F(x) &= P(X \le x) = \int_{-\infty}^{x} \lambda e^{-\lambda y}\, dy \\
&= \int_{0}^{x} \lambda e^{-\lambda y}\, dy \\
&= \left(-e^{-\lambda y}\right)\Big|_{y=0}^{x} \\
&= -e^{-\lambda x} - \left(-e^{0}\right) \\
&= -e^{-\lambda x} + 1 \\
&= 1 - e^{-\lambda x}
\end{aligned}
$$

Note Card

$$F(x) = 1 - e^{-\lambda x}$$

Problem 4-79

4-79. The time to failure (in hours) of fans in a personal computer can be modeled by an exponential distribution with $\lambda = 0.0003$.

(a) What proportion of the fans will last at least 10,000 hours?

(b) What proportion of the fans will last at most 7000 hours?

$$P(X > 10{,}000)$$

$$= 1 - F(10{,}000)$$

$$= 1 - \left(1 - e^{-.0003(10{,}000)}\right)$$

$$= e^{-3} = \cancel{.05}\ .04979$$

image

Question 1 (1 point)

Listen

Let X be an exponential random variable with lambda=14.007, find the value x such that P(X > x)= 0.423.

- 0.0027
- 0.2499
- 0.9973
- 0.0614
- 0.0393

$$P(X > x) = 1 - F(x) = .423$$

$$1 - (1 - e^{-\lambda x}) = .423$$

$$e^{-\lambda x} = .423$$

$$e^{-14.007x} = .423$$

$$\ln(e^{-14.007x}) = \ln .423$$

$$-14.007x = \ln .423$$

$$x = \frac{-\ln .423}{14.007} = .06143$$

Question 3 (1 point)

Listen

Let X be an exponential random variable, with lambda=14.141, find P(X>0.0501).

- ~~5.963~~
- ~~6.963~~
- 0.4924
- 0.5076

$$P(X > .0501) = 1 - F(.0501)$$

$$= 1 - \left(1 - e^{-14.141(.0501)}\right)$$

$$= e^{-14.141(.0501)}$$

$$= e^{-.70846} = .49240$$

Question 5 (1 point)

Listen

Let X be an exponential random variable with lambda=44.609, find P(X<0.0051).

- 0.7965
- 35.5319
- 0.2035
- -34.5319

$$F(.0051) = 1 - e^{-\lambda x}$$

$$= 1 - e^{-44.609(.0051)}$$

$$= 1 - e^{-.22751}$$

$$= .20348$$

Question 7 (1 point)

Listen

Let X be an exponential random variable with lambda=47.378, find P(0.0415 <X< 0.0613).

- 0.9452
- 0.0852
- -4.0366
- 0.14

$$= F(.0613) - F(.0415)$$

$$= 1 - e^{-47.378(.0613)} - \left(1 - e^{-47.378(.0415)}\right)$$

$$= -e^{-2.90427} + e^{-1.96619}$$

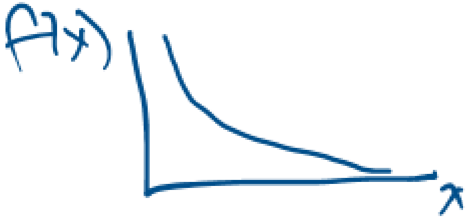
$$= .0852$$

Question 9 (1 point)

Listen

Let X be an exponential random variable with lambda=49.187. Find the value x such that P(X< x)= 0.638.

- 1.0
- 0.0
- 3.0E-4
- 0.0207
- 0.0091



$$F(x) = .638$$

$$1 - e^{-49.187x} = .638$$

$$+e^{-49.187x} = .362$$

$$\ln\left(e^{-49.187x}\right) = \ln .362$$

$$-49.187x = \ln .362$$

$$x = \frac{-\ln .362}{49.187} = .02066$$

Lack of Memory Property

- The mathematical definition is

$$P(X < t_1 + t_2 | X > t_1) = P(X < t_2)$$

- That is "the probability of a failure time that is less than $t_1 + t_2$ given the failure time is greater than t_1 is the probability that the item's failure time is less than t_2
- This property is unique to the exponential distribution
- Often used to model the reliability of electronic components.

Problem 4–80

4-80. The time between the arrival of electronic messages at your computer is exponentially distributed with a mean of two hours.

(a) What is the probability that you do not receive a message during a two-hour period?

(b) If you have not had a message in the last four hours, what is the probability that you do not receive a message in the next two hours?

(c) What is the expected time between your fifth and sixth messages?

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Relationship to the Poisson Distribution

- Let Y be a Poisson random variable with parameter λ. Note: Y represents the number of occurrences per unit
- Let X be a random variable that records the time between occurrences for the same process as Y
- X has an exponential distribution with parameter λ

Lognormal Distribution

impurities

- Let W have a normal distribution with mean θ and variance ω^2; then $X = \exp(W)$ is a **lognormal random variable** with pdf

$$f(x) = \frac{1}{x\omega\sqrt{2\pi}} \exp\left[-\frac{(\ln(x) - \theta)^2}{2\omega^2}\right] \quad 0 < x < \infty$$

- The mean of X is

$$E(X) = e^{\theta + \omega^2/2}$$

- The variance of X is

$$V(X) = e^{2\theta + \omega^2}\left(e^{\omega^2} - 1\right)$$

Example Problem

3-47. Suppose that X has a lognormal distribution with parameters $\theta = 5$ and $\omega^2 = 9$. Determine the following:

(a) $P(X < 13{,}300)$

(b) The value for x such that $P(X \le x) = 0.95$

(c) The mean and variance of X Montgomery, Runger and Hubble

image

Gamma Distribution

Γ - Gamma

- The random variable X with pdf

$$f(x) = \frac{\lambda^r x^{r-1} e^{-\lambda x}}{\Gamma(r)}, \quad \text{for } x > 0$$

is a **gamma random variable** with parameters $\lambda > 0$ and $r > 0$.

- The gamma function is

$$\Gamma(r) = \int_0^{\infty} x^{r-1} e^{-x}\, dx \quad \text{for } r > 0$$

with special properties:

- $\Gamma(r)$ is finite
- $\Gamma(r) = (r-1)\Gamma(r-1)$
- For any positive integer r, $\Gamma(r) = (r-1)!$
  - $\Gamma(1/2) = \pi^{1/2}$

Gamma Distribution

- The mean and variance are

$$\mu = E(X) = r/\lambda \text{ and } \sigma^2 = V(X) = r/\lambda^2$$

- We will not work any probability problems using the gamma distribution

Gamma Tables

Gamma Function

n	$\Gamma(n)$	n	$\Gamma(n)$	n	$\Gamma(n)$	n	$\Gamma(n)$
0.0100	99.4327	0.5100	1.7384	1.0100	0.9943	1.5100	0.8866
0.0200	49.4423	0.5200	1.7058	1.0200	0.9888	1.5200	0.8870
0.0300	32.7850	0.5300	1.6747	1.0300	0.9836	1.5300	0.8876
0.0400	24.4610	0.5400	1.6448	1.0400	0.9784	1.5400	0.8882
0.0500	19.4701	0.5500	1.6161	1.0500	0.9735	1.5500	0.8889
0.0600	16.1457	0.5600	1.5886	1.0600	0.9687	1.5600	0.8896
0.0700	13.7736	0.5700	1.5623	1.0700	0.9642	1.5700	0.8905
0.0800	11.9966	0.5800	1.5369	1.0800	0.9597	1.5800	0.8914
0.0900	10.6162	0.5900	1.5126	1.0900	0.9555	1.5900	0.8924
0.1000	9.5135	0.6000	1.4892	1.1000	0.9513	1.6000	0.8935
0.1100	8.6127	0.6100	1.4667	1.1100	0.9474	1.6100	0.8947
0.1200	7.8632	0.6200	1.4450	1.1200	0.9436	1.6200	0.8959
0.1300	7.2302	0.6300	1.4242	1.1300	0.9399	1.6300	0.8972
0.1400	6.6887	0.6400	1.4041	1.1400	0.9364	1.6400	0.8986
0.1500	6.2203	0.6500	1.3848	1.1500	0.9330	1.6500	0.9001
0.1600	5.8113	0.6600	1.3662	1.1600	0.9298	1.6600	0.9017
0.1700	5.4512	0.6700	1.3482	1.1700	0.9267	1.6700	0.9033
0.1800	5.1318	0.6800	1.3309	1.1800	0.9237	1.6800	0.9050
0.1900	4.8468	0.6900	1.3142	1.1900	0.9209	1.6900	0.9068
0.2000	4.5908	0.7000	1.2981	1.2000	0.9182	1.7000	0.9086
0.2100	4.3599	0.7100	1.2825	1.2100	0.9156	1.7100	0.9106
0.2200	4.1505	0.7200	1.2675	1.2200	0.9131	1.7200	0.9126
0.2300	3.9598	0.7300	1.2530	1.2300	0.9108	1.7300	0.9147
0.2400	3.7855	0.7400	1.2390	1.2400	0.9085	1.7400	0.9168
0.2500	3.6256	0.7500	1.2254	1.2500	0.9064	1.7500	0.9191
0.2600	3.4785	0.7600	1.2123	1.2600	0.9044	1.7600	0.9214
0.2700	3.3426	0.7700	1.1997	1.2700	0.9025	1.7700	0.9238
0.2800	3.2169	0.7800	1.1875	1.2800	0.9007	1.7800	0.9262
0.2900	3.1001	0.7900	1.1757	1.2900	0.8990	1.7900	0.9288
0.3000	2.9916	0.8000	1.1642	1.3000	0.8975	1.8000	0.9314
0.3100	2.8903	0.8100	1.1532	1.3100	0.8960	1.8100	0.9341
0.3200	2.7958	0.8200	1.1425	1.3200	0.8946	1.8200	0.9368
0.3300	2.7072	0.8300	1.1322	1.3300	0.8934	1.8300	0.9397
0.3400	2.6242	0.8400	1.1222	1.3400	0.8922	1.8400	0.9426
0.3500	2.5461	0.8500	1.1125	1.3500	0.8912	1.8500	0.9456
0.3600	2.4727	0.8600	1.1031	1.3600	0.8902	1.8600	0.9487
0.3700	2.4036	0.8700	1.0941	1.3700	0.8893	1.8700	0.9518
0.3800	2.3383	0.8800	1.0853	1.3800	0.8885	1.8800	0.9551
0.3900	2.2765	0.8900	1.0768	1.3900	0.8879	1.8900	0.9584
0.4000	2.2182	0.9000	1.0686	1.4000	0.8873	1.9000	0.9618
0.4100	2.1628	0.9100	1.0607	1.4100	0.8868	1.9100	0.9652
0.4200	2.1104	0.9200	1.0530	1.4200	0.8864	1.9200	0.9688
0.4300	2.0605	0.9300	1.0456	1.4300	0.8860	1.9300	0.9724
0.4400	2.0132	0.9400	1.0384	1.4400	0.8858	1.9400	0.9761
0.4500	1.9681	0.9500	1.0315	1.4500	0.8857	1.9500	0.9799
0.4600	1.9252	0.9600	1.0247	1.4600	0.8856	1.9600	0.9837
0.4700	1.8843	0.9700	1.0182	1.4700	0.8856	1.9700	0.9877
0.4800	1.8453	0.9800	1.0119	1.4800	0.8857	1.9800	0.9917
0.4900	1.8080	0.9900	1.0059	1.4900	0.8859	1.9900	0.9958
0.5000	1.7725	1.0000	1.0000	1.5000	0.8862	2.0000	1.0000

image

on midterm

Weibull Distribution

- The random variable X with pdf

$$f(x) = \frac{\beta}{\delta}\left(\frac{x}{\delta}\right)^{\beta-1} \exp\left[-\left(\frac{x}{\delta}\right)^{\beta}\right], \quad \text{for } x > 0$$

is a **Weibull random variable** with scale parameter $\delta > 0$ and shape parameter $\beta > 0$

- The CDF for the Weibull distribution is

$$F(x) = 1 - \exp\left[-\left(\frac{x}{\delta}\right)^{\beta}\right]$$

- The mean of the Weibull distribution is

$$\mu = E(X) = \delta\Gamma\left(1 + \frac{1}{\beta}\right)$$

- The variance of the Weibull distribution is

$$\sigma^2 = V(X) = \delta^2\Gamma\left(1 + \frac{2}{\beta}\right) - \delta^2\left[\Gamma\left(1 + \frac{1}{\beta}\right)\right]^2$$

Weibull Problem

Part a) $P(X < 100)$

$= F(100)$

$= 1 - \exp\left[-\left(\frac{x}{\delta}\right)^{\beta}\right]$

$= 1 - \exp\left[-\left(\frac{100}{125}\right)^{5}\right]$

$= .27941$

45. Suppose that fracture strength (MPa) of silicon nitride braze joints under certain conditions has a Weibull distribution with $\beta = 5$ and $\delta = 125$ (suggested by data in the article "Heat-Resistant Active Brazing of Silicon Nitride: Mechanical Evaluation of Braze Joints," (*Welding J.*, August 1997: 300s–304s).

a. What proportion of such joints have a fracture strength of at most 100? Between 100 and 150?

b. What strength value separates the weakest 50% of all joints from the strongest 50%?

c. What strength value characterizes the weakest 5% of all joints?

image

Find x such that $P(X<x) = .05$

$$F(x) = .05$$

$$1 - \exp\left[-\left(\frac{x}{125}\right)^5\right] = .05$$

$$+\exp\left[-\left(\frac{x}{125}\right)^5\right] = +.95$$

$$\ln\left[\exp\left\{-\left(\frac{x}{125}\right)^5\right\}\right] = \ln .95$$

$$-\left(\frac{x}{125}\right)^5 = \ln .95$$

$$\left(\frac{x}{125}\right)^5 = -\ln .95$$

$$\left(\frac{x}{125}\right)^5 = -\ln .95$$

$$\left(\frac{x}{125}\right) = (-\ln .95)^{\frac{1}{5}}$$

$$x = 125(-\ln .95)^{\frac{1}{5}}$$

$$= 69.01161$$

There are also 2 common ways to write Weibull distribution exponential

$$F(x) = 1 - e^{-\lambda x}$$

Weibull

$$F(x) = 1 - e^{-\left(\frac{x}{\delta}\right)^\beta}$$

Let $\beta = 1$

$$F(x) = 1 - e^{-\left(\frac{x}{\delta}\right)}$$

observe $\lambda = \frac{1}{\delta}$

exponential is a special case of Weibull with $\beta = 1$

Reliability – Bathtub Curve