Attendance: 1~13

MANE 3332.05

Lecture 14

Agenda

- Start Part Two of Course
- Major Quiz Announcement
  - All Part One Quizzes will be modified to two attempts and highest grade counted
  - Deadline for completing Part One Quizzes is 10/23/2025 ~~12:30~~ PM 3:30pm
- Normal Quiz (assigned 10/14/2025, due 10/16/2026)
- Exponential Practice Problems (assigned 10/14/2025, due 10/16/2025)
- Exponential Quiz (assigned 10/16/2025, due 10/23/2025)
- Schedule
- Attendance
- Questions?

Handouts

- Lecture 14 slides (Powerpoint)
- Lecture 14 slides - marked (pdf)

Class Schedule

Tuesday Date and Topic(s)	Thursday Date and Topic(s)
10/14: Exponential and Weibull distributions	**10/16:** Chapter 5 (not on midterm)
10/21: Midterm Review	**10/23:** Midterm Exam

Chapter Five

- Joint Probability Distributions
- Contains eight sections
- We will only examine 5.4 (Covariance and Correlation) and 5.6 (linear functions of random variables)

Covariance and Correlation

Covariance

- When two or more variables are defined on a probability space, it is useful to describe how they vary together
- A common measure of the relationship between two random variables is the **covariance**

$$\sigma_{XY} \quad = \quad E(XY) - \mu_X \mu_Y$$

Covariance, continued

- Theoretically for two continuous random variables with joint probability distribution function $f_{XY}(x, y)$, the covariance is found by

continuous

$$\sigma_{XY} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\,y f_{XY}(x, y)\,dx\,dy - \mu_X\mu_Y$$

Discrete

$$\sigma_{xy} = \sum_{\text{all } x}\sum_{\text{all } y} xy\, f_{xy}(x,y) - \mu_x \mu_y$$

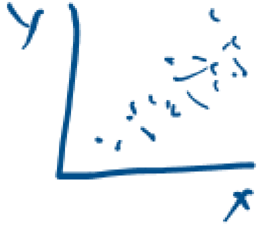
$-\infty < \sigma_{xy} < \infty$

Covariance and Independence

- If X and Y are independent random variables,

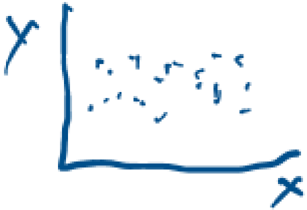
$$\sigma_{XY} = 0$$

- However, $\sigma_{XY} = 0$ does not imply that X and Y are independent. Textbook mentions Figure 5-13(d)
- *SPECIAL CASE.* IF X and Y are normal random variables and have $\sigma_{XY} = 0$, then X and Y are independent



$\sigma_{xy} > 0$

$\sigma_{xy} < 0$



$\sigma_{xy} \approx 0.0$

circle

$\sigma_{xy} = 0$

Zero Covariance but

not independent

Sample Covariance

- To calculate the sample covariance use

$$s_{XY} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$$

- Easily done in software

ρ - rho

Correlation

- The correlation between two random variables X and Y is

$$\rho_{XY} = \frac{\text{cov}(X, Y)}{\sqrt{V(X)V(Y)}} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

- For any two random variables X and Y

$$-1 \leq \rho_{XY} \leq 1$$

- If X and Y are independent $\rho_{XY} = 0$. The converse is not true.

Sample Correlation Coefficient

- To calculate the sample correlation coefficient,

$$r_{XY} = \frac{s_{XY}}{\sqrt{S_X^2 S_Y^2}}$$

Summary

- Correlation is a linear measure and will not work for non-linear relationships
- Correlation is a measure of association; it does not prove cause and effect relationships

  -Examine examples at Spurious Correlations website

Linear Functions of Random Variables

Functions of Random Variables

- Additive System. Let X be a random variable with mean μ and variance σ^2. Define a new random variable Y

$$Y = X + c$$

$E(X+c) = E(X) + E(c)$

It follows that

$$E(Y) = E(X) + c = \mu + c$$
$$V(Y) = V(X) + 0 = \sigma^2$$

$V(X+c) = V(X) + V(\cancel{c}) = \sigma^2$

0

Linear Functions of Random Variables

Functions of Random Variables

- Multiplicative System. Consider the new random variable Y

$$Y = cX$$

It follows that

$$\begin{aligned} E(Y) &= E(cX) = cE(X) = c\mu \\ V(Y) &= V(cX) = c^2V(x) = c^2\sigma^2 \end{aligned}$$

c^2

$$\bar{X} = \frac{1}{n} \sum X_i = \sum_{i=1}^{n} \frac{1}{n} X_i \rightarrow c_i = \frac{1}{n}$$

Linear Combination

Case for independent random variables

- A **linear combination** of the random variables $X_1, X_2, \ldots, X_n$ is
$$Y = c_1 X_1 + c_2 X_2 + \cdots + c_n X_n$$
- The mean of a linear combination of random variables is
$$E(Y) = c_1 \mu_1 + c_2 \mu_2 + \cdots + c_n \mu_n$$
- The variance of a linear combination of random variables is
$$V(Y) = c_1^2 \sigma_1^2 + c_2^2 \sigma_2^2 + \cdots + c_n^2 \sigma_n^2$$

$$\bar{X} = \sum_{i=1}^{n} \frac{1}{n} X_i$$

$$E(\bar{X}) = E\left(\sum_{i=1}^{n} \frac{1}{n} X_i\right) = \sum_{i=1}^{n} E\left(\frac{1}{n} X_i\right)$$

$$= \frac{1}{n}\left[\mu \cdot n\right] = \mu$$

$$V(\bar{X}) = V\left(\sum_{i=1}^{n} \frac{1}{n} X_i\right) = \sum_{i=1}^{n} V\left(\frac{1}{n} X_i\right)$$

$$= \frac{1}{n^2}\left[\sum V(X_i)\right]$$

$$= \frac{1}{n^2}\left[n\sigma^2\right] = \frac{\sigma^2}{n}$$

Linear Combination of Non-independent R.V.

Let $X_1, X_2, \dots, X_n$ be random variables with means $E(X_i) = \mu_i$, variances $V(X_i) = \sigma_i^2$ and covariances $\text{Cov}(X_i, X_j)$ for $i, j = 1, 2, \dots, n$ with $i < j$

- The linear combination is defined to be

$$Y = c_1 X_1 + c_2 X_2 + \cdots + c_n X_n$$

- The mean of Y is

$$E(Y) = c_1 \mu_1 + c_2 \mu_2 + \cdots + c_n \mu_n$$

same as the independent case

Linear Combination of Non-independent R.V.

- and the variance is

$$
\begin{aligned}
V(Y) \quad &= \quad c_1^2\sigma_1^2 + c_2^2\sigma_2^2 + \cdots + c_n^2\sigma_n^2 \\
&\qquad +2\sum\sum_{i<j} c_i c_j \mathrm{Cov}\left(X_i, X_j\right)
\end{aligned}
$$

Linear Combination Problem

$E(5X_1 + 10X_2)$

$= 5E(X_1) + 10E(X_2)$

$= 5(2) + 10(4) = 50$

$V(5X_1 + 10X_2)$

$= 5^2 V(X_1) + 10^2 V(X_2)$

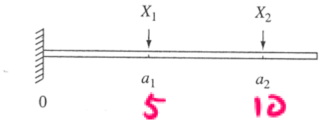
$= 5^2(.5)^2 + 10^2(1)^2$

$= 106.25$

$\sigma = \sqrt{106.25} = 10.31$

Source: Devore (2000) Prob & Statistics

66. If two loads are applied to a cantilever beam as shown in the accompanying drawing, the bending moment at 0 due to the loads is $a_1X_1 + a_2X_2$.



Standard deviation

σ

a. Suppose that X_1 and X_2 are independent rv's with means 2 and 4 kips, respectively, and standard deviations .5 and 1.0 kip, respectively. If $a_1 = 5$ ft and $a_2 = 10$ ft, what is the expected bending moment and what is the standard deviation of the bending moment?

b. If X_1 and X_2 are normally distributed, what is the probability that the bending moment will exceed 75 kip-ft?

$P(x > 75)$

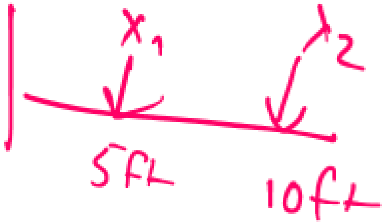
c. Suppose the positions of the two loads are random variables. Denoting them by A_1 and A_2, assume that these variables have means of 5 and 10 ft, respectively, that each has a standard deviation of .5, and that all A_i's and X_i's are independent of one another. What is the expected moment now?

d. For the situation of part (c), what is the variance of the bending moment?

e. If the situation is as described in part (a) except that $\text{Corr}(X_1, X_2) = .5$ (so that the two loads are not independent), what is the variance of the bending moment?

linear combination problem

e. If the situation is as described in part (a) except that $\mathrm{Corr}(X_1, X_2) = .5$ (so that the two loads are not independent), what is the variance of the bending moment?



$$V(\cancel{X_1+X_2}\ 5X_1+10X_2) = c_1^2 V(X_1) + c_2^2 V(X_2) + 2c_1c_2 \mathrm{Cov}(X_1, X_2)$$

$$\mathrm{Corr}(X_1, X_2) = \frac{\mathrm{Cov}(X_1, X_2)}{\sqrt{V(X_1)\,V(X_2)}} = 0.5$$

$$\frac{\mathrm{Cov}(X_1, X_2)}{\sqrt{(.5)^2(1)^2}} = .5$$

$$\mathrm{Cov}(X_1, X_2) = .5(.5) = .25$$

$$V(5X_1 + 10Y_2) = 5^2 V(X_1) + 10^2 V(X_2) + 2(5)(10)(.25)$$

$$= 5^2(.5)^2 + 10^2(1)^2 + 2(5)(10)(.25)$$

$$= 131.25$$

Linear Combination Practice Problems

Central Limit Theorem

If $X_1, X_2, \ldots, X_n$ is a random sample of size n taken from a population with mean μ and variance σ^2, and if $\overline{X}$ is the sample mean, the limiting form of the distribution of

$$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}}$$

as $n \to \infty$, is the standard normal distribution

- Incredibly useful theorem
- See example below
- n often does not have to be very large
  - If the population is continuous, unimodal and symmetric, often n can be as small as 4 or 5
  - Larger samples will be required in other situations
  - If $n \geq 30$ the normal approximation will work satisfactorily regardless of the shape of the population

CLT Illustration

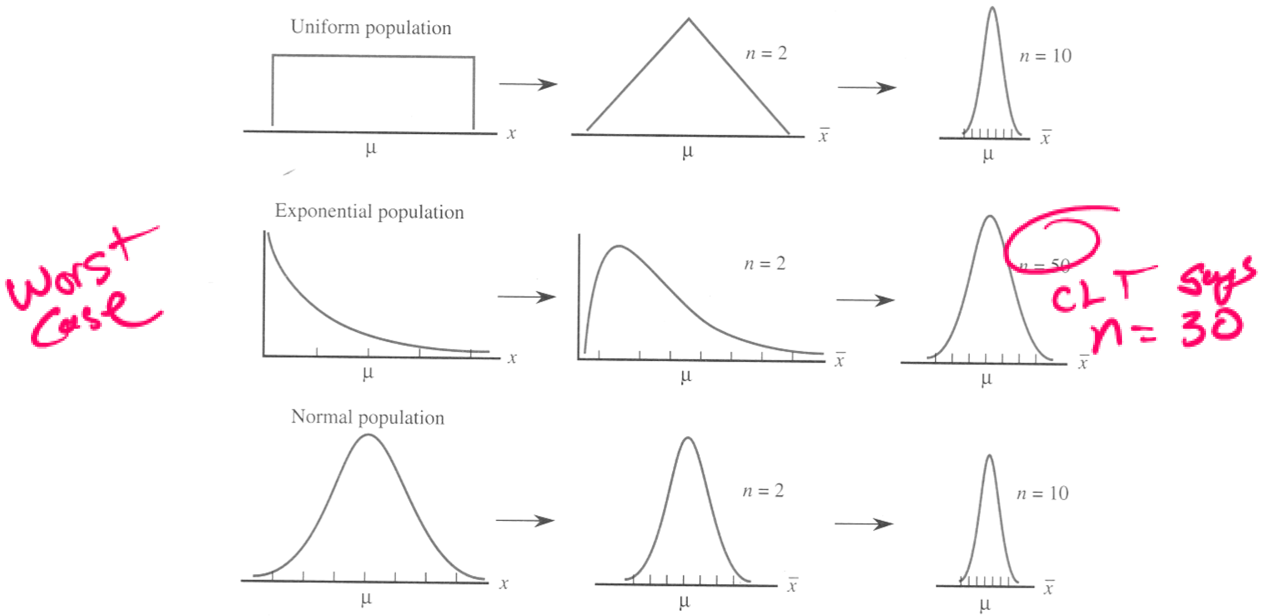


Figure 5.18 The Central Limit Theorem: The sampling distribution of $\bar{x}$ approaches a normal distribution as the sample size n increases.

central limit theorem illustration