MANE 3332.05

LECTURE 9

Agenda

- Complete Chapter 3 lectures
- Start Chapter 4 lectures
- Binomial Quiz (assigned 9/25/2025, due 9/30/2025)
- Poisson Practice Problems (assigned 9/30/2025, due 10/2/2025)
- Schedule

Handouts

- Lecture 9 Slides - Powerpoint
- Lecture 9 Slides - marked (pdf)

Tuesday Date and Topic(s)	Thursday Date and Topic(s)
9/30: Poisson Distribution, Chapter 4	**10/2:** standard normal
10/7: normal distribution	**10/9:** Exponential and Weibull distributions
10/14: Chapter 5 (not on midterm)	**10/16:** Midterm Review
10/21: Midterm Exam	**10/23:** Continue Part Two

Poisson Process

- The number of events over an interval (such as time) is a discrete random variable that is often modelled by the Poisson distribution
- The length of the interval between events is often modeled by the (continuous) exponential distribution
- These two distributions are related

Poisson Process

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Poisson Process

Assume that the events occur at random throughout the interval. If the interval can be partitioned into subintervals of small enough length such that

1. The probability of more than one count in a subinterval is zero
2. The probability of one count in a subinterval is the same for all subintervals and proportional to the length of the subinterval, and
3. The count in each subinterval is independent of other subintervals, the random experiment is called a *Poisson process*

Poisson Distribution λ - lambda

If the mean number of counts in the interval is $\lambda > 0$, the random variable X that equals the number of counts in the interval has a **Poisson distribution** with parameter λ

- The Poisson PMF is

$$f(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \quad x = 0,1,2,\dots \quad ?$$

- The mean of a Poisson random variable is

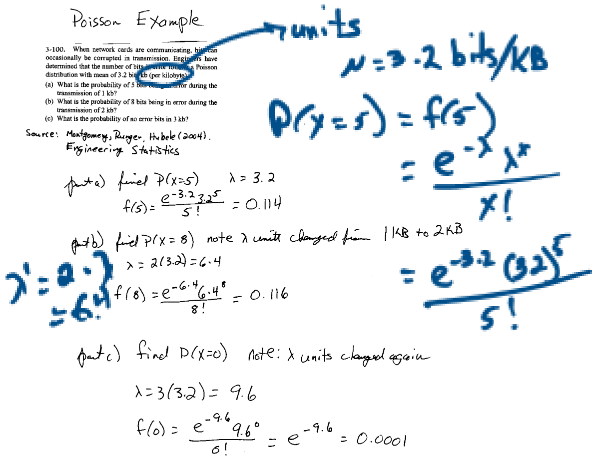
$$E(X) = \mu = \lambda$$

- The variance of a Poisson random variable is

$$V(X) = \sigma^2 = \lambda$$

unique

Poisson Example



Poisson Example

3-100. When network cards are communicating, bits can occasionally be corrupted in transmission. Engineers have determined that the number of bits in error follows a Poisson distribution with mean of 3.2 bits/kb (per kilobyte).

(a) What is the probability of 5 bits being in error during the transmission of 1 kb?
(b) What is the probability of 8 bits being in error during the transmission of 2 kb?
(c) What is the probability of no error bits in 3 kb?

Source: Montgomery, Runger, Hubele (2004). Engineering Statistics

units

$\lambda = 3.2$ bits/KB

$$P(x=5) = f(5) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-3.2}(3.2)^5}{5!}$$

part a) find $P(X=5)$ $\lambda = 3.2$

$$f(5) = \frac{e^{-3.2}3.2^5}{5!} = 0.114$$

part b) find $P(X=8)$ note λ units changed from 1 KB to 2 KB

$\lambda' = 2 \cdot 3.2 = 6.4$

$\lambda = 2(3.2) = 6.4$

$$f(8) = \frac{e^{-6.4}6.4^8}{8!} = 0.116$$

part c) find $P(X=0)$ note: λ units changed again

$\lambda = 3(3.2) = 9.6$

$$f(0) = \frac{e^{-9.6}9.6^0}{0!} = e^{-9.6} = 0.0001$$

image

Question 1 (1 point)

Listen

Let X be a Poisson random variable with with parameter: lambda=1.309. Find P(X<=0).

1) 0.2701

2) 0.0

3) 0.7299

4) The correct answer is not provided.

5) 0.3161

6) 1.0

$P(X \leq 0) = f(\emptyset)$

$X \in \{0, 1, 2, \dots\}$

Poisson $x = 0, 1, 2, \dots$

$$f(\emptyset) = \frac{e^{-1.309}(1.309)^{\emptyset}}{\emptyset!} = 0.27009$$

$.2701$

Question 3 (1 point)

Listen

Let X be a Poisson random variable with with parameter: lambda=2.546. Find P(X<2).

1) 0.5321

2) The correct answer is not provided.

3) 0.278

4) 0.4679

5) 0.2541

6) 0.722

$P(X<2) = f(0) + f(1)$

$f(0) = \dfrac{e^{-2.546}(2.546)^0}{0!} = .07839$

$f(1) = \dfrac{e^{-2.546}(2.546)^1}{1!} = .19959$

$P(X<2) = .07839 + .19959 = .27799$

For Poisson $x = 0, 1, 2, \dots$

$$P(X \le 0) = f(0) + \underbrace{\cancel{f(-1)}}_{0} + \underbrace{\cancel{f(-2)}}_{0} + \dots$$

$$P(X < 0) = 0$$

Question 5 (1 point)

Listen

Let X be a Poisson random variable with with parameter: lambda=2.754. Find P(X>3).

$$P(X>3) = f(4) + f(5) + f(6) + \dots$$

$$= 1 - \{f(0) + f(1) + f(2) + f(3)\}$$

$$= 1 - F(3)$$

1) 0.5195

2) The correct answer is not provided.

3) 0.2217

4) 0.2978

5) 0.4805

6) 0.7022

Question 7 (1 point)

$$\frac{e^{-\lambda}\lambda^{x}}{x!}$$

Listen

Let X be a Poisson random variable with parameter: lambda = 4.085. Find P(X=1).

$P(X=1) = f(1)$

1) 0.0855

2) The correct answer is not provided.

3) 0.9145

$$f(1) = \frac{e^{-4.085}(4.085)^{1}}{1!}$$

4) 0.0168

5) 1.0

$= .06872$

6) 0.9832

Question 9 (1 point)

Listen

Let X be a Poisson random variable with with parameter: lambda=2.249. Find P(X>=1).

$$P(X \geq 1) = f(1) + f(2) + \cdots$$

1) 0.3428

2) 0.6572

$$= 1 - f(0)$$

3) 0.1055

$$f(0) = \frac{e^{-2.249} \, 2.249^0}{0!} = e^{-2.249} = .1055$$

4) 1.0

5) 0.2373

6) The correct answer is not provided.

$$P(X \geq 1) = 1 - .1055$$

$$= .8945$$

Poisson Practice Problems

CHAPTER 4 CONTENT

Continuous Random Variable

- The **probability distribution** of a random variable X is a description of the set of probabilities associated with the possible values of X → real-valued
- Density functions are commonly used in engineering to describe physical systems.
- A **probability density function** $f(x)$ can be used to describe the probability distribution of a continuous random variable

pdf

area under the curve

Probability Density Function

- Notice the difference from a discrete random variable
- The formal definition of a probability density function is a function such that

1. $f(x) \geq 0$
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$
3. $P(a \leq X \leq b) = \int_a^b f(x)\,dx$

Chapter 3

1) $f(x) \geq 0$

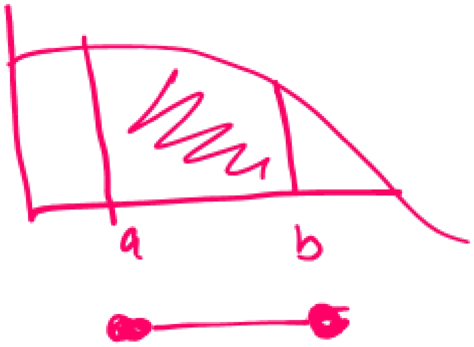
2) $\sum_{\text{all } x} f(x) = 1$

Most Confusing slide in the Course

Probability Density Function

- Any interesting property of continuous random variables is
$$P(x_1 \le X \le x_2) = P(x_1 < X \le x_2) = P(x_1 \le X < x_2)$$
$$= P(x_1 < X < x_2)$$
- Does not apply to discrete random variables
- Explanation

$x \geq a \quad X < b$



$X \geq a \quad X \leq b$

Cumulative Distribution Function

The cumulative distribution function for a continuous random variable X is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f\,(y)\,dy$$

Mean and Variance of a Continuous Random Variable

- The mean value of a continuous random variable is defined to be

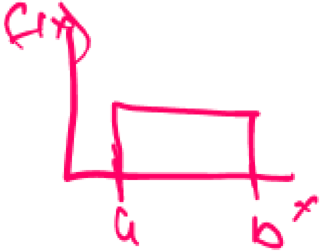
$$\mu = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

- The variance of a continuous random variable is defined to be

$$\begin{aligned} \sigma^2 &= V(X) = E(X-\mu)^2 \\ &= \int_{-\infty}^{\infty} (x-\mu)^2\, f(x)\, dx = \int_{-\infty}^{\infty} x^2\, f(x)\, dx - \mu^2 \end{aligned}$$

- The standard deviation of X is

$$\sigma = \sqrt{V(X)}$$



Continuous Uniform Distribution

The continuous uniform distribution is the analog of the discrete uniform distribution in that all outcomes are equally likely to occur

- A continuous uniform distribution for the random variable X has a probability density function

$$f(x) = \frac{1}{b-a}, \quad a \le x \le b$$

- The mean of the uniform distribution is

$$\mu = E(X) = \frac{a+b}{2}$$

- The variance of X is

$$\sigma^2 = V(X) = \frac{(b-a)^2}{12}$$

$$\mu = \int_{-\infty}^{\infty} x\, f(x)\, dx \qquad f(x) = \frac{1}{b-a} \text{ for } a \le x \le b$$

$$= \int_{-\infty}^{\infty} x \left(\frac{1}{b-a}\right) dx = \left(\frac{1}{b-a}\right) \int_{-\infty}^{\infty} x\, dx$$

$$= \left(\frac{1}{b-a}\right) \left\{ \cancel{\int_{-\infty}^{a} x\, dx} + \int_{a}^{b} x\, dx + \cancel{\int_{b}^{\infty} x\, dx} \right\} \qquad \emptyset$$

$$= \left(\frac{1}{b-a}\right) \left(\frac{x^2}{2}\right)\Bigg|_{x=a}^{b}$$

$$= \left(\frac{1}{b-a}\right) \left(\frac{b^2-a^2}{2}\right) = \left(\frac{1}{\cancel{b-a}}\right) \frac{\cancel{(b-a)}(b+a)}{2}$$

$$= \frac{a+b}{2}$$

CDF - Uniform $\quad f(y) = \frac{1}{b-a}$

for $a \leq y \leq b$

$$\int_{-\infty}^{x} f(y)\,dy$$

$$= \int_{a}^{x} \frac{1}{b-a}\,dy = \frac{1}{b-a} y \Big|_{y=a}^{x}$$

$$= \frac{1}{b-a}(x-a)$$

$$= \frac{x-a}{b-a}$$

$f(x) = \frac{1}{b-a}$

$f(x) = 2$ for $49.75 \le x \le 50.25$

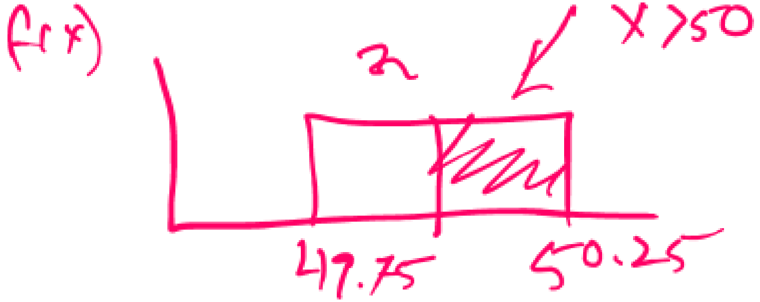
Uniform Problem 4.1.6

- See page P-25

$$P(x>50) = \int_{50}^{50.25} 2\,dx$$

$$= 2x \Big|_{x=50}^{50.25}$$

$$= 2(50.25-50) = 1/2$$



$$P(X>50) = 1 - F(50)$$

$$F(x) = \frac{x-a}{b-a} = \frac{x-49.75}{50.25-49.75} = 2(x-49.75)$$

$$1 - F(50) = 1 - 2(50-49.75)$$

$$= 0.5$$

The Normal Distribution

- The normal distribution is the most widely used and important distribution in statistics.
- You must master this!
- A random variable X with probability density function

$$-\infty < x < \infty f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \text{for}$$

has a **normal distribution** with parameters μ and σ where $-\infty < \mu < \infty$ and $\sigma > 0$

- The normal distribution with parameters μ and σ is denoted $N(\mu, \sigma^2)$
- An interesting web-site is http://www.seeingstatistics.com/seeingTour/normal/shape3.html

Attendance 1-C

Mean and Variance of the Normal Distribution

- The mean of the normal distribution with parameters μ and σ is

$$E(X) = \mu$$

- The variance of the normal distribution with parameters μ and σ is

$$V(X) = \sigma^2$$

Central Limit Theorem

- Brief introduction
- States that the distribution of the average of independent random variables will tend towards a normal distribution as n gets large
- More details later

Calculating Normal Probabilities

- Is somewhat complicated
- The difficulty is $\int_a^b f\,(x)\,dx$ does not have a closed form solution
- Probabilities must be found by numerical techniques (tabled values)
- It is very helpful to draw a sketch of the desired probabilities (I require this)

The Standard Normal Distribution

- A normal random variable with $\mu = 0$ and $\sigma = 1$ is called a **standard normal** random variable
- A standard normal random variable is denoted as z
- The cumulative distribution function for a standard normal is defined to be the function

$$\Phi(z) = P(Z \leq z)$$

- These probabilities are contained in Appendix Table III on pages A-8 and A-9

Cumulative Standard Normal Distribution

$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\, du$$

TABLE III Cumulative Standard Normal Distribution

z	−0.09	−0.08	−0.07	−0.06	−0.05	−0.04	−0.03	−0.02	−0.01	−0.00
−3.9	0.000033	0.000034	0.000036	0.000037	0.000039	0.000041	0.000042	0.000044	0.000046	0.000048
−3.8	0.000050	0.000052	0.000054	0.000057	0.000059	0.000062	0.000064	0.000067	0.000069	0.000072
−3.7	0.000075	0.000078	0.000082	0.000085	0.000088	0.000092	0.000096	0.000100	0.000104	0.000108
−3.6	0.000112	0.000117	0.000121	0.000126	0.000131	0.000136	0.000142	0.000147	0.000153	0.000159
−3.5	0.000165	0.000172	0.000179	0.000185	0.000193	0.000200	0.000208	0.000216	0.000224	0.000233
−3.4	0.000242	0.000251	0.000260	0.000270	0.000280	0.000291	0.000302	0.000313	0.000325	0.000337
−3.3	0.000350	0.000362	0.000376	0.000390	0.000404	0.000419	0.000434	0.000450	0.000467	0.000483
−3.2	0.000501	0.000519	0.000538	0.000557	0.000577	0.000598	0.000619	0.000641	0.000664	0.000687
−3.1	0.000711	0.000736	0.000762	0.000789	0.000816	0.000845	0.000874	0.000904	0.000935	0.000968
−3.0	0.001001	0.001035	0.001070	0.001107	0.001144	0.001183	0.001223	0.001264	0.001306	0.001350
−2.9	0.001395	0.001441	0.001489	0.001538	0.001589	0.001641	0.001695	0.001750	0.001807	0.001866
−2.8	0.001926	0.001988	0.002052	0.002118	0.002186	0.002256	0.002327	0.002401	0.002477	0.002555
−2.7	0.002635	0.002718	0.002803	0.002890	0.002980	0.003072	0.003167	0.003264	0.003364	0.003467
−2.6	0.003573	0.003681	0.003793	0.003907	0.004025	0.004145	0.004269	0.004396	0.004527	0.004661
−2.5	0.004799	0.004940	0.005085	0.005234	0.005386	0.005543	0.005703	0.005868	0.006037	0.006210
−2.4	0.006387	0.006569	0.006756	0.006947	0.007143	0.007344	0.007549	0.007760	0.007976	0.008198
−2.3	0.008424	0.008656	0.008894	0.009137	0.009387	0.009642	0.009903	0.010170	0.010444	0.010724
−2.2	0.011011	0.011304	0.011604	0.011911	0.012224	0.012545	0.012874	0.013209	0.013553	0.013903
−2.1	0.014262	0.014629	0.015003	0.015386	0.015778	0.016177	0.016586	0.017003	0.017429	0.017864
−2.0	0.018309	0.018763	0.019226	0.019699	0.020182	0.020675	0.021178	0.021692	0.022216	0.022750
−1.9	0.023295	0.023852	0.024419	0.024998	0.025588	0.026190	0.026803	0.027429	0.028067	0.028717
−1.8	0.029379	0.030054	0.030742	0.031443	0.032157	0.032884	0.033625	0.034379	0.035148	0.035930
−1.7	0.036727	0.037538	0.038364	0.039204	0.040059	0.040929	0.041815	0.042716	0.043633	0.044565
−1.6	0.045514	0.046479	0.047460	0.048457	0.049471	0.050503	0.051551	0.052616	0.053699	0.054799
−1.5	0.055917	0.057053	0.058208	0.059380	0.060571	0.061780	0.063008	0.064256	0.065522	0.066807
−1.4	0.068112	0.069437	0.070781	0.072145	0.073529	0.074934	0.076359	0.077804	0.079270	0.080757
−1.3	0.082264	0.083793	0.085343	0.086915	0.088508	0.090123	0.091759	0.093418	0.095098	0.096801
−1.2	0.098525	0.100273	0.102042	0.103835	0.105650	0.107488	0.109349	0.111233	0.113140	0.115070
−1.1	0.117023	0.119000	0.121001	0.123024	0.125072	0.127143	0.129238	0.131357	0.133500	0.135666
−1.0	0.137857	0.140071	0.142310	0.144572	0.146859	0.149170	0.151505	0.153864	0.156248	0.158655
−0.9	0.161087	0.163543	0.166023	0.168528	0.171056	0.173609	0.176185	0.178786	0.181411	0.184060
−0.8	0.186733	0.189430	0.192150	0.194894	0.197662	0.200454	0.203269	0.206108	0.208970	0.211855
−0.7	0.214764	0.217695	0.220650	0.223627	0.226627	0.229650	0.232695	0.235762	0.238852	0.241964
−0.6	0.245097	0.248252	0.251429	0.254627	0.257846	0.261086	0.264347	0.267629	0.270931	0.274253
−0.5	0.277595	0.280957	0.284339	0.287740	0.291160	0.294599	0.298056	0.301532	0.305026	0.308538
−0.4	0.312067	0.315614	0.319178	0.322758	0.326355	0.329969	0.333598	0.337243	0.340903	0.344578
−0.3	0.348268	0.351973	0.355691	0.359424	0.363169	0.366928	0.370700	0.374484	0.378281	0.382089
−0.2	0.385908	0.389739	0.393580	0.397432	0.401294	0.405165	0.409046	0.412936	0.416834	0.420740
−0.1	0.424655	0.428576	0.432505	0.436441	0.440382	0.444330	0.448283	0.452242	0.456205	0.460172
0.0	0.464144	0.468119	0.472097	0.476078	0.480061	0.484047	0.488033	0.492022	0.496011	0.500000

Cumulative Standard Normal Distribution

$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\, du$$

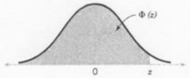


TABLE III Cumulative Standard Normal Distribution (*continued*)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.500000	0.503989	0.507978	0.511967	0.515953	0.519939	0.532922	0.527903	0.531881	0.535856
0.1	0.539828	0.543795	0.547758	0.551717	0.555760	0.559618	0.563559	0.567495	0.571424	0.575345
0.2	0.579260	0.583166	0.587064	0.590954	0.594835	0.598706	0.602568	0.606420	0.610261	0.614092
0.3	0.617911	0.621719	0.625516	0.629300	0.633072	0.636831	0.640576	0.644309	0.648027	0.651732
0.4	0.655422	0.659097	0.662757	0.666402	0.670031	0.673645	0.677242	0.680822	0.684386	0.687933
0.5	0.691462	0.694974	0.698468	0.701944	0.705401	0.708840	0.712260	0.715661	0.719043	0.722405
0.6	0.725747	0.729069	0.732371	0.735653	0.738914	0.742154	0.745373	0.748571	0.751748	0.754903
0.7	0.758036	0.761148	0.764238	0.767305	0.770350	0.773373	0.776373	0.779350	0.782305	0.785236
0.8	0.788145	0.791030	0.793892	0.796731	0.799546	0.802338	0.805106	0.807850	0.810570	0.813267
0.9	0.815940	0.818589	0.821214	0.823815	0.826391	0.828944	0.831472	0.833977	0.836457	0.838913
1.0	0.841345	0.843752	0.846136	0.848495	0.850830	0.853141	0.855428	0.857690	0.859929	0.862143
1.1	0.864334	0.866500	0.868643	0.870762	0.872857	0.874928	0.876976	0.878999	0.881000	0.882977
1.2	0.884930	0.886860	0.888767	0.890651	0.892512	0.894350	0.896165	0.897958	0.899727	0.901475
1.3	0.903199	0.904902	0.906582	0.908241	0.909877	0.911492	0.913085	0.914657	0.916207	0.917736
1.4	0.919243	0.920730	0.922196	0.923641	0.925066	0.926471	0.927855	0.929219	0.930563	0.931888
1.5	0.933193	0.934478	0.935744	0.936992	0.938220	0.939429	0.940620	0.941792	0.942947	0.944083
1.6	0.945201	0.946301	0.947384	0.948449	0.949497	0.950529	0.951543	0.952540	0.953521	0.954486
1.7	0.955435	0.956367	0.957284	0.958185	0.959071	0.959941	0.960796	0.961636	0.962462	0.963273
1.8	0.964070	0.964852	0.965621	0.966375	0.967116	0.967843	0.968557	0.969258	0.969946	0.970621
1.9	0.971283	0.971933	0.972571	0.973197	0.973810	0.974412	0.975002	0.975581	0.976148	0.976705
2.0	0.977250	0.977784	0.978308	0.978822	0.979325	0.979818	0.980301	0.980774	0.981237	0.981691
2.1	0.982136	0.982571	0.982997	0.983414	0.983823	0.984222	0.984614	0.984997	0.985371	0.985738
2.2	0.986097	0.986447	0.986791	0.987126	0.987455	0.987776	0.988089	0.988396	0.988696	0.988989
2.3	0.989276	0.989556	0.989830	0.990097	0.990358	0.990613	0.990863	0.991106	0.991344	0.991576
2.4	0.991802	0.992024	0.992240	0.992451	0.992656	0.992857	0.993053	0.993244	0.993431	0.993613
2.5	0.993790	0.993963	0.994132	0.994297	0.994457	0.994614	0.994766	0.994915	0.995060	0.995201
2.6	0.995339	0.995473	0.995604	0.995731	0.995855	0.995975	0.996093	0.996207	0.996319	0.996427
2.7	0.996533	0.996636	0.996736	0.996833	0.996928	0.997020	0.997110	0.997197	0.997282	0.997365
2.8	0.997445	0.997523	0.997599	0.997673	0.997744	0.997814	0.997882	0.997948	0.998012	0.998074
2.9	0.998134	0.998193	0.998250	0.998305	0.998359	0.998411	0.998462	0.998511	0.998559	0.998605
3.0	0.998650	0.998694	0.998736	0.998777	0.998817	0.998856	0.998893	0.998930	0.998965	0.998999
3.1	0.999032	0.999065	0.999096	0.999126	0.999155	0.999184	0.999211	0.999238	0.999264	0.999289
3.2	0.999313	0.999336	0.999359	0.999381	0.999402	0.999423	0.999443	0.999462	0.999481	0.999499
3.3	0.999517	0.999533	0.999550	0.999566	0.999581	0.999596	0.999610	0.999624	0.999638	0.999650
3.4	0.999663	0.999675	0.999687	0.999698	0.999709	0.999720	0.999730	0.999740	0.999749	0.999758
3.5	0.999767	0.999776	0.999784	0.999792	0.999800	0.999807	0.999815	0.999821	0.999828	0.999835
3.6	0.999841	0.999847	0.999853	0.999858	0.999864	0.999869	0.999874	0.999879	0.999883	0.999888
3.7	0.999892	0.999896	0.999900	0.999904	0.999908	0.999912	0.999915	0.999918	0.999922	0.999925
3.8	0.999928	0.999931	0.999933	0.999936	0.999938	0.999941	0.999943	0.999946	0.999948	0.999950
3.9	0.999952	0.999954	0.999956	0.999958	0.999959	0.999961	0.999963	0.999964	0.999966	0.999967

Standard Normal Problem

5.1.1 Suppose that $Z \sim N(0, 1)$. Find:

(a) $P(Z \leq 1.34)$

(b) $P(Z \geq -0.22)$

(c) $P(-2.19 \leq Z \leq 0.43)$

(d) $P(0.09 \leq Z \leq 1.76)$

(e) $P(|Z| \leq 0.38)$

(f) The value of x for which $P(Z \leq x) = 0.55$

(g) The value of x for which $P(Z \geq x) = 0.72$

(h) The value of x for which $P(|Z| \leq x) = 0.31$

image

Standard Normal Practice Problems

Standardizing (the z-transform)

- Suppose X is a normal random variable with mean μ and variance σ^2

$$P(X \leq x) = P\left(\frac{X - \mu}{\sigma} \leq \frac{x - \mu}{\sigma}\right) = P(Z \leq z)$$

- The z-value is $z = (x - \mu)/\sigma$
- Result allows the standard normal tables to be used to calculate probabilities for any normal distribution

Normal Probability Problem

5.1.3 Suppose that $X \sim N(10, 2)$. Find:

(a) $P(X \leq 10.34)$

(b) $P(X \geq 11.98)$

(c) $P(7.67 \leq X \leq 9.90)$

(d) $P(10.88 \leq X \leq 13.22)$

(e) $P(|X - 10| \leq 3)$

(f) The value of x for which $P(X \leq x) = 0.81$

(g) The value of x for which $P(X \geq x) = 0.04$

(h) The value of x for which $P(|X - 10| \geq x) = 0.63$

image

Normal Practice Problems

Normal Approximation to the Binomial Distribution

- If X is a binomial random variable,

$$Z = \frac{X - np}{\sqrt{np(1-p)}}$$

is approximately a standard normal random variable. Consequently, probabilities computed from Z can be used to approximate probabilities for X

- Usually holds when

$$np > 5 \quad \text{and } n(1-p) > 5$$

- How good are the approximations?

Problem

4. A process yields 10% defective items. If 100 items are randomly selected from the process, what is the probability that the number of defectives
(a) exceeds 13?
(b) is less than 8?

Source: Walpole, Myers, Myers & Ye

image

Normal Approximation - Figure

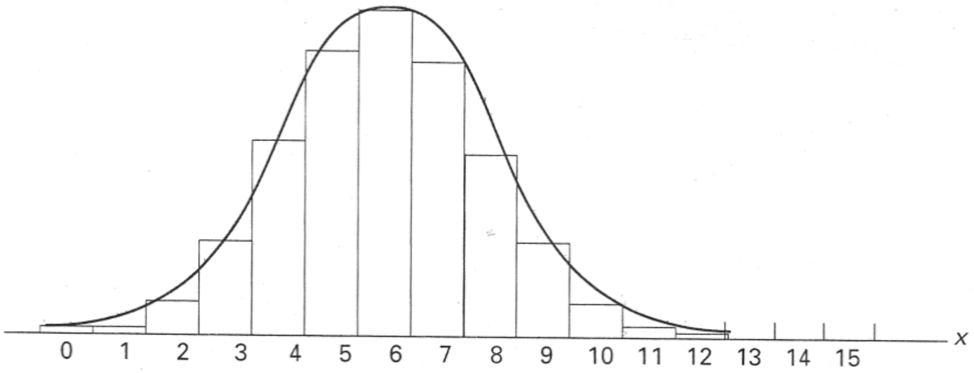


Figure 6.22 Normal approximation of $b(x; 15, 0.4)$.

Source: Walpole, Myers, Myers & Ye

image

Rework Problem using Continuity Correction Factor

- Are the approximations improved?

Normal Approximation to the Poisson Distribution

- If X is a Poisson random variable with $E(X) = \lambda$ and $V(X) = \lambda$,

$$Z = \frac{X - \lambda}{\sqrt{\lambda}}$$

is approximately a standard normal random variable.

Exponential Distribution

- The exponential distribution is widely used in the area of reliability and life-test data.
- Ostle, et. al. (1996) list the following applications of the exponential distribution
  - the number of feet between two consecutive erroneous records on a computer tape,
  - the lifetime of a component of a particular device,
  - the length of a life of a radioactive material and
  - the time to the next customer service call at a service desk

Exponential Distribution

- The PDF for an exponential distribution with parameter $\lambda > 0$ is

$$f(x) = \lambda e^{-\lambda x}, \quad \text{for } 0 \leq x < \infty$$

- The mean of X is

$$\mu = E(X) = \frac{1}{\lambda}$$

- The variance of X is

$$\sigma^2 = V(X) = \frac{1}{\lambda^2}$$

Note that other authors define $f(x) = \frac{1}{\theta} e^{-x/\theta}$. Either definition is acceptable. However one must be aware of which definition is being used.

The Exponential CDF

The CDF for the exponential distribution is easy to derive

$$
\begin{aligned}
F(x) &= P(X \leq x) = \int_{-\infty}^{x} \lambda\, e^{-\lambda y}\, dy \\
&= \int_{0}^{x} \lambda\, e^{-\lambda y}\, dy \\
&= \left(-e^{-\lambda y}\right)\Big|_{y=0}^{x} \\
&= -e^{-\lambda x} - (-e^{0}) \\
&= -e^{-\lambda x} + 1 \\
&= 1 - e^{-\lambda x}
\end{aligned}
$$

Problem 4-79

4-79. The time to failure (in hours) of fans in a personal computer can be modeled by an exponential distribution with $\lambda = 0.0003$.

(a) What proportion of the fans will last at least 10,000 hours?

(b) What proportion of the fans will last at most 7000 hours?

image

Lack of Memory Property

- The mathematical definition is
$$P(X < t_1 + t_2 | X > t_1) = P(X < t_2)$$
- That is “the probability of a failure time that is less than $t_1 + t_2$ given the failure time is greater than t_1 is the probability that the item’s failure time is less than t_2
- This property is unique to the exponential distribution
- Often used to model the reliability of electronic components.

Problem 4–80

4-80. The time between the arrival of electronic messages at your computer is exponentially distributed with a mean of two hours.

(a) What is the probability that you do not receive a message during a two-hour period?

(b) If you have not had a message in the last four hours, what is the probability that you do not receive a message in the next two hours?

(c) What is the expected time between your fifth and sixth messages?

image

Relationship to the Poisson Distribution

- Let Y be a Poisson random variable with parameter λ. Note: Y represents the number of occurrences per unit
- Let X be a random variable that records the time between occurrences for the same process as Y
- X has an exponential distribution with parameter λ

Lognormal Distribution

- Let W have a normal distribution with mean θ and variance ω^2; then $X = \exp(W)$ is a **lognormal random variable** with pdf

$$f(x) = \frac{1}{x\omega\sqrt{2\pi}} \exp\left[-\frac{(\ln(x) - \theta)^2}{2\omega^2}\right] \quad 0 < x < \infty$$

- The mean of X is

$$E(X) = e^{\theta + \omega^2/2}$$

- The variance of X is

$$V(X) = e^{2\theta + \omega^2}\left(e^{\omega^2} - 1\right)$$

Example Problem

3-47. Suppose that X has a lognormal distribution with parameters $\theta = 5$ and $\omega^2 = 9$. Determine the following:

(a) $P(X < 13{,}300)$

(b) The value for x such that $P(X \leq x) = 0.95$

(c) The mean and variance of X Montgomery, Runger, Hubble

image

Gamma Distribution

- The random variable X with pdf

$$f(x) = \frac{\lambda^r x^{r-1} e^{-\lambda x}}{\Gamma(r)}, \quad \text{for } x > 0$$

is a **gamma random variable** with parameters $\lambda > 0$ and $r > 0$.

- The gamma function is

$$\Gamma(r) = \int_0^\infty x^{r-1}\, e^{-x}\, dx \quad \text{for } r > 0$$

with special properties:

- $\Gamma(r)$ is finite
- $\Gamma(r) = (r-1)\Gamma(r-1)$
- For any positive integer r, $\Gamma(r) = (r-1)!$
  - $\Gamma(1/2) = \pi^{1/2}$

Gamma Distribution

- The mean and variance are

$$\mu = E(X) = r/\lambda \text{ and } \sigma^2 = V(X) = r/\lambda^2$$

- We will not work any probability problems using the gamma distribution

Gamma Tables

Gamma Function

n	$\Gamma(n)$	n	$\Gamma(n)$	n	$\Gamma(n)$	n	$\Gamma(n)$
0.0100	99.4327	0.5100	1.7384	1.0100	0.9943	1.5100	0.8866
0.0200	49.4423	0.5200	1.7058	1.0200	0.9888	1.5200	0.8870
0.0300	32.7850	0.5300	1.6747	1.0300	0.9836	1.5300	0.8876
0.0400	24.4610	0.5400	1.6448	1.0400	0.9784	1.5400	0.8882
0.0500	19.4701	0.5500	1.6161	1.0500	0.9735	1.5500	0.8889
0.0600	16.1457	0.5600	1.5886	1.0600	0.9687	1.5600	0.8896
0.0700	13.7736	0.5700	1.5623	1.0700	0.9642	1.5700	0.8905
0.0800	11.9966	0.5800	1.5369	1.0800	0.9597	1.5800	0.8914
0.0900	10.6162	0.5900	1.5126	1.0900	0.9555	1.5900	0.8924
0.1000	9.5135	0.6000	1.4892	1.1000	0.9513	1.6000	0.8935
0.1100	8.6127	0.6100	1.4667	1.1100	0.9474	1.6100	0.8947
0.1200	7.8632	0.6200	1.4450	1.1200	0.9436	1.6200	0.8959
0.1300	7.2302	0.6300	1.4242	1.1300	0.9399	1.6300	0.8972
0.1400	6.6887	0.6400	1.4041	1.1400	0.9364	1.6400	0.8986
0.1500	6.2203	0.6500	1.3848	1.1500	0.9330	1.6500	0.9001
0.1600	5.8113	0.6600	1.3662	1.1600	0.9298	1.6600	0.9017
0.1700	5.4512	0.6700	1.3482	1.1700	0.9267	1.6700	0.9033
0.1800	5.1318	0.6800	1.3309	1.1800	0.9237	1.6800	0.9050
0.1900	4.8468	0.6900	1.3142	1.1900	0.9209	1.6900	0.9068
0.2000	4.5908	0.7000	1.2981	1.2000	0.9182	1.7000	0.9086
0.2100	4.3599	0.7100	1.2825	1.2100	0.9156	1.7100	0.9106
0.2200	4.1505	0.7200	1.2675	1.2200	0.9131	1.7200	0.9126
0.2300	3.9598	0.7300	1.2530	1.2300	0.9108	1.7300	0.9147
0.2400	3.7855	0.7400	1.2390	1.2400	0.9085	1.7400	0.9168
0.2500	3.6256	0.7500	1.2254	1.2500	0.9064	1.7500	0.9191
0.2600	3.4785	0.7600	1.2123	1.2600	0.9044	1.7600	0.9214
0.2700	3.3426	0.7700	1.1997	1.2700	0.9025	1.7700	0.9238
0.2800	3.2169	0.7800	1.1875	1.2800	0.9007	1.7800	0.9262
0.2900	3.1001	0.7900	1.1757	1.2900	0.8990	1.7900	0.9288
0.3000	2.9916	0.8000	1.1642	1.3000	0.8975	1.8000	0.9314
0.3100	2.8903	0.8100	1.1532	1.3100	0.8960	1.8100	0.9341
0.3200	2.7958	0.8200	1.1425	1.3200	0.8946	1.8200	0.9368
0.3300	2.7072	0.8300	1.1322	1.3300	0.8934	1.8300	0.9397
0.3400	2.6242	0.8400	1.1222	1.3400	0.8922	1.8400	0.9426
0.3500	2.5461	0.8500	1.1125	1.3500	0.8912	1.8500	0.9456
0.3600	2.4727	0.8600	1.1031	1.3600	0.8902	1.8600	0.9487
0.3700	2.4036	0.8700	1.0941	1.3700	0.8893	1.8700	0.9518
0.3800	2.3383	0.8800	1.0853	1.3800	0.8885	1.8800	0.9551
0.3900	2.2765	0.8900	1.0768	1.3900	0.8879	1.8900	0.9584
0.4000	2.2182	0.9000	1.0686	1.4000	0.8873	1.9000	0.9618
0.4100	2.1628	0.9100	1.0607	1.4100	0.8868	1.9100	0.9652
0.4200	2.1104	0.9200	1.0530	1.4200	0.8864	1.9200	0.9688
0.4300	2.0605	0.9300	1.0456	1.4300	0.8860	1.9300	0.9724
0.4400	2.0132	0.9400	1.0384	1.4400	0.8858	1.9400	0.9761
0.4500	1.9681	0.9500	1.0315	1.4500	0.8857	1.9500	0.9799
0.4600	1.9252	0.9600	1.0247	1.4600	0.8856	1.9600	0.9837
0.4700	1.8843	0.9700	1.0182	1.4700	0.8856	1.9700	0.9877
0.4800	1.8453	0.9800	1.0119	1.4800	0.8857	1.9800	0.9917
0.4900	1.8080	0.9900	1.0059	1.4900	0.8859	1.9900	0.9958
0.5000	1.7725	1.0000	1.0000	1.5000	0.8862	2.0000	1.0000

image

Weibull Distribution

- The random variable X with pdf

$$f(x) = \frac{\beta}{\delta}\left(\frac{x}{\delta}\right)^{\beta-1} \exp\left[-\left(\frac{x}{\delta}\right)^{\beta}\right], \quad \text{for } x > 0$$

is a **Weibull random variable** with scale parameter $\delta > 0$ and shape parameter $\beta > 0$

- The CDF for the Weibull distribution is

$$F(x) = 1 - \exp\left[-\left(\frac{x}{\delta}\right)^{\beta}\right]$$

- The mean of the Weibull distribution is

$$\mu = E(X) = \delta\Gamma\left(1 + \frac{1}{\beta}\right)$$

- The variance of the Weibull distribution is

$$\sigma^2 = V(X) = \delta^2\Gamma\left(1 + \frac{2}{\beta}\right) - \delta^2\left[\Gamma\left(1 + \frac{1}{\beta}\right)\right]^2$$

Weibull Problem

45. Suppose that fracture strength (MPa) of silicon nitride braze joints under certain conditions has a Weibull distribution with $\beta = 5$ and $\delta = 125$ (suggested by data in the article "Heat-Resistant Active Brazing of Silicon Nitride: Mechanical Evaluation of Braze Joints," (*Welding J.*, August 1997: 300s–304s).

a. What proportion of such joints have a fracture strength of at most 100? Between 100 and 150?

b. What strength value separates the weakest 50% of all joints from the strongest 50%?

c. What strength value characterizes the weakest 5% of all joints?

image

Weibull Practice Problems