# MANE 3351

# Lecture 23

**Classroom Management**

**Agenda**

- Gauss Jordan Elimination, Partial Pivoting, Matrix Inversion using Gauss Jordan
- Homework 7 (assigned 11/24/25, due 12/1/25 - no late submissions)
- Lab today for those who did not finish

## Resources

**Handouts**

Lecture 23 slides

Lecture 23 slides marked

**Calendar**

| Week | Monday Lecture | Wednesday Lecture |
|---|---|---|
| 13 | **11/24:** Gauss-Jordan Elimination | **11/26:** no class |
| 14 | **12/1:** Lecture 24 | **12/3:** Lecture 25 |
| 15 | **12/8:** Lecture 26 | **12/10:** Lecture 27 |

**Final Exam is Monday 12/15/2025 8:00 - 9:45 AM**

I will be off-campus in an ABET visit and a proctor will be arranged for the final exam.

**Assignments**

- Homework 6 (assigned 11/17, due 11/24)
- Lab 9 (assigned 11/19/25, due 11/26/25 (before lab))
- Homework 7 (assigned 11/40, due 12/1 - no late submissions)

# Gauss-Jordan Elimination

- A Step-by-Step Method for Solving Linear Systems
- An extension of row-echelon form

$$\begin{pmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{pmatrix} \rightarrow \left(\begin{array}{cc|c} 1 & ? & ? \\ 0 & 1 & ? \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & ? & ? & ? \\ 0 & 1 & ? & ? \\ 0 & 0 & 1 & ? \end{array}\right)$$

(1) main diagonal elements are all 1's

(2) any column values below main diagonal are equal to 0

## Introduction to Gauss-Jordan Elimination

- Gauss-Jordan elimination is a method to solve systems of linear equations.
- Goal: Transform the matrix into reduced row-echelon form (RREF).
- Key feature: The solution is read directly from the matrix.
- Distinguish from Row Echelon Form

$$\begin{pmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{pmatrix} \rightarrow \left(I \,\vdots\, \cdots \right) \rightarrow \left(\begin{array}{cc|c} 1 & 0 & b_1 \\ 0 & 1 & b_2 \end{array}\right)$$

$$y = b_2$$

$$x = b_1$$

**Key Steps in Gauss-Jordan Elimination**

1. **Row Operations**:
   - Swap two rows.
   - Multiply a row by a nonzero scalar.
   - Add or subtract multiples of one row to another.
2. **Transform to RREF**:
   - Each leading entry is 1 (pivot).
   - Pivots are the only nonzero entries in their column.

## Example Problem

RREF/Gauss-Jordan

Solve the system:

$$x + y + z = 6$$
$$2x + 3y + z = 14$$
$$y + 2z = 8$$

Pivot Row

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 2 & 3 & 1 & 14 \\ 0 & 1 & 2 & 8 \end{pmatrix} \xrightarrow{R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & 2 & 8 \end{pmatrix} \xrightarrow{R_1 - R_2,\; R_3 - R_2} \begin{pmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 3 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 3 & 6 \end{pmatrix} \xrightarrow{\frac{1}{3}R_3} \begin{pmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$R_1 - 2R_3$

$R_2 + R_3$

$$\begin{pmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{array}{l} x = 0 \\ y = 4 \\ z = 2 \end{array}$$

→ Pivoting

**Step-by-Step Solution**

1. Convert the system to an augmented matrix.

2. Use row operations to make the first pivot 1.

3. Eliminate all other entries in the pivot column.

4. Repeat for each subsequent pivot.

**Final Result**

- Matrix in reduced row-echelon form (RREF):

$$\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{bmatrix}$$

- Solution: $x = a, y = b, z = c$.

**Applications of Gauss-Jordan Elimination**

- Solving systems of linear equations.
- Finding inverse matrices. $\rightarrow (A \mid I) \rightarrow (I \mid A^{-1})$
- Used in engineering, physics, computer science, etc.

**Summary of Gauss-Jordan Elimination**

- Gauss-Jordan elimination simplifies solving linear systems.
- Three row operations ensure clarity and consistency.
- Matrix RREF provides direct solutions.

# Partial Pivoting

- A **pivot** is the leading non-zero element in a row used to simplify the matrix.

- Pivoting ensures numerical stability during elimination.

- Types of pivoting:
  1. **Partial Pivoting:** Swap rows to place the largest absolute value in the pivot position.

  2. **Complete Pivoting:** Reorder rows and columns to place the largest value in the pivot position.

**Why Pivoting is Necessary**

- Avoid division by small numbers (reduce round-off errors).
- Improve accuracy in solving systems.
- Ensure numerical stability for ill-conditioned matrices.

## Example Problem

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = b$$

Solve the system of equations:

$$0.0001x + y + z = 1$$
$$x + y + z = 6$$
$$2x + y + 10z = 20$$

$$\begin{pmatrix} .0001 & 1 & 1 & 1 \\ 1 & 1 & 1 & 6 \\ 2 & 1 & 10 & 20 \end{pmatrix}$$

## Form the Augmented Matrix

Represent the system as an augmented matrix:

$$\begin{bmatrix} 0.0001 & 1 & 1 & | & 1 \\ 1 & 1 & 1 & | & 6 \\ 2 & 1 & 10 & | & 20 \end{bmatrix}$$

- The first three columns are the coefficients of the variables $x, y, z$.
- The fourth column is the constants on the right-hand side.

**Perform Partial Pivoting**

1. Compare the absolute values in the first column:

$$\text{Column 1: } |0.0001|, |1|, |2|$$

Largest value: 2.

2. Swap row 1 with row 3:

$$\begin{bmatrix} 2 & 1 & 10 & | & 20 \\ 1 & 1 & 1 & | & 6 \\ 0.0001 & 1 & 1 & | & 1 \end{bmatrix}$$

## Elimination Step-by-Step

1. Eliminate entries below the pivot in the first column:

$$R_2 \to R_2 - \frac{1}{2} R_1, \quad R_3 \to R_3 - \frac{0.0001}{2} R_1$$

2. Resulting matrix:

fixed ✓ for column 2

$$\begin{bmatrix} 2 & 1 & 10 & | & 20 \\ 0 & 0.5 & -4 & | & -4 \\ 0 & 0.99995 & 0.9995 & | & 0.9995 \end{bmatrix}$$

3. Repeat pivoting and elimination for columns 2 and 3.

① for column 2, swap rows 2 & 3 because |.99995| > |.5|

② no more partial pivoting required for column 3

**Slide 8: Complete Solution**

- After performing Gauss-Jordan elimination with pivoting:

$$\begin{bmatrix} 1 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & 1 \end{bmatrix}$$

- Solution:

$$x = 2, \quad y = 1, \quad z = 1$$

**Slide 9: Challenges and Limitations**

- Pivoting increases computational complexity for larger matrices.
- Requires additional bookkeeping for row swaps.
- May not guarantee exact accuracy for ill-conditioned matrices.

**Applications of Pivoting**

- Engineering: Solving large systems of linear equations.
- Computer science: Numerical simulations and optimizations.
- Physics: Stability in solving differential equations.

## Summary

- Pivoting improves the accuracy and stability of Gauss-Jordan elimination.

almost always

- Partial pivoting is a practical and efficient choice.

- Numerical stability is crucial for solving large or complex systems.

# Using Gauss-Jordan Elimination to find Inverse Matrix

- To find the inverse of a matrix $A$, we augment it with the identity matrix $I$.

- Perform Gauss-Jordan elimination to transform $A$ into $I$, turning $I$ into $A^{-1}$.

- Works only if the matrix $A$ is invertible (determinant $\neq 0$).

**Key Steps**

1. Write the augmented matrix $[A|I]$.

2. Use row operations to transform $A$ into the identity matrix $I$.

3. The resulting augmented matrix will be $[I|A^{-1}]$.

**Example Problem**

Find the inverse of:

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$$

- Form augmented matrix:

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 5 & 3 & 0 & 1 \end{array}\right]$$

**Step-by-Step Solution**

1. Scale the first row to make the pivot 1:

$$R_1 \to R_1/2$$

Result:

$$\begin{bmatrix} 1 & 0.5 & | & 0.5 & 0 \\ 5 & 3 & | & 0 & 1 \end{bmatrix}$$

2. Eliminate the first column of $R_2$:

$$R_2 \rightarrow R_2 - 5 \cdot R_1$$

Result:

$$\begin{bmatrix} 1 & 0.5 & | & 0.5 & 0 \\ 0 & 0.5 & | & -2.5 & 1 \end{bmatrix}$$

3. Scale the second row to make the pivot 1:

$$R_2 \rightarrow R_2/0.5$$

Result:

$$\begin{bmatrix} 1 & 0.5 & | & 0.5 & 0 \\ 0 & 1 & | & -5 & 2 \end{bmatrix}$$

4. Eliminate the second column of $R_1$:

$$R_1 \rightarrow R_1 - 0.5 \cdot R_2$$

Result:

$$\begin{bmatrix} 1 & 0 & | & 3 & -1 \\ 0 & 1 & | & -5 & 2 \end{bmatrix}$$

$I_2$

$A^{-1}$

$A^{-1}A = AA^{-1} = I$

$$A^{-1}A = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 3\cdot 2-(1)(5) & 3(1)+(-1)(3) \\ -5(2)+(2)(5) & -5(1)+2(3) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

## Final Result

- The final augmented matrix is:

$$\left[\begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 1 & -5 & 2 \end{array}\right]$$

$$A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$$

$$|A| = 2\cdot 3 - (1)(5) = 1$$

- Inverse of $A$:

$$A^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} = \frac{1}{1}\begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$$

can find inverse

**Conditions for Invertibility**

- A matrix $A$ is invertible if:

1. It is square ($n \times n$).

2. Determinant of $A \neq 0$.

- For $2 \times 2$:

$$\det(A) = ad - bc \neq 0$$

## Summary

- Gauss-Jordan elimination transforms $A$ into $I$, revealing $A^{-1}$.
- Method highlights the importance of row operations.
- Verifiable through matrix multiplication: $A \cdot A^{-1} = I$.

# Discussion of ChatGPT

- These slides were prepared using ChatGPT
  - The partial pivoting slides contained an error that required a prompt to be modified