MANE 3351

Lecture 7

Classroom Management

Agenda

- Taylor Series Expansion
- Homework 2
- Schedule
- Lab session at 9:30

Resources

Handouts

- Lecture 7 Slides
- Lecture 7 Marked Slides

Assignments

- Homework 1 (assigned 9/22/2025, due 9/29/2025 (before 11:59 PM)
- Homework 2 (assigned 9/24/2025, due 10/1/2025 (before 9:30 AM - no late submissions)
- Lab 3 (assigned 9/24/2025, due 10/1/2025 (before 9:30 AM))
- Read textbook pages 1 - 16

Schedule

Lecture/Lab	Date	Topic
7	9/24	Taylor Series, Homework 2 (due 10/1 - no late work), Lab 3 (due 10/1)
8	9/29	Roots of Equations, bisection method (not on Test 1)
9	10/1	Bisection Method Error Analysis, False Position (not on Test 1)
10	10/6	Test 1 (lectures 1-7)

Lecture Content

- Taylor Series Expansion

Taylor Series

Introduction to Taylor Series

Cheney and Kincaid [^1] provide some commonly used Taylor series.

How big should k be?

Taylor Series

→ Already Solved

Familiar (and useful) examples of Taylor series are the following:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad (|x| < \infty) \tag{1}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \qquad (|x| < \infty) \tag{2}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \qquad (|x| < \infty) \tag{3}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{k=0}^{\infty} x^k \qquad (|x| < 1) \tag{4}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k} \qquad (-1 < x \leqq 1) \tag{5}$$

Taylor Series

Example

To find e^8, recall $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

k

0 1. $e^8 = 1$

k = 1

1 2. $e^8 = 1 + x = 1 + 8$

2 3. $e^8 = 1 + 8 + \frac{x^2}{2!} = 1 + 8 + \frac{64}{2}$

3 4. $e^8 = 1 + 8 + \frac{64}{2} + \frac{x^3}{3!} = 1 + 8 + \frac{64}{2} + \frac{512}{6}$

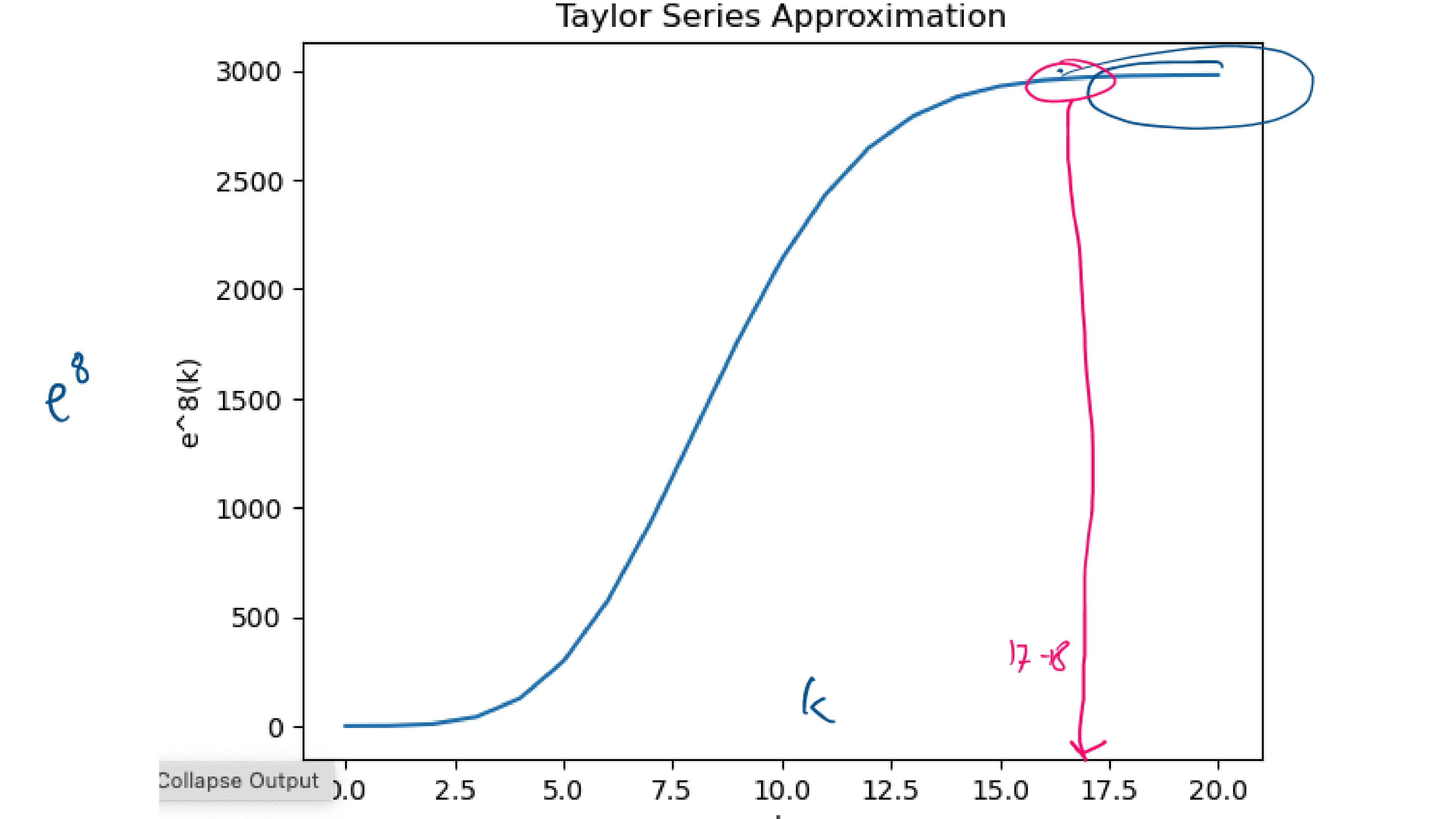
Python Code for Jupyter Notebook

```
import math
import numpy as np
import matplotlib.pyplot as plt
def eTaylor(x,k):
    y=0.0
    for i in range(k):
        #print(i)
        y=y+(x**i)/math.factorial(i)
        #print("i=",i," y=",y," i!= ",math.factorial(i)," x^i=",x**i)
    return y
k=np.arange(21)
print(k)
print(k[0])
y=0.0*k
for i in np.nditer(k):
    #print(i)
    y[i]=eTaylor(8,i)
#print(y)
#plotting code
fig, ax = plt.subplots()
ax.plot(k, y)
ax.set(xlabel='k', ylabel='e^8(k)',
       title='Taylor Series Approximation')
plt.show()
```

function

$\frac{x^i}{i!}$

k is an array



Taylor Series Approximation
e^8(k)
3000
2500
2000
1500
1000
500
0
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
Collapse Output

Source: textbook, page 14

Taylor Series Expansion about a Point

$f^{(j)}(x_0) \rightarrow j^{th}$ derivative of $f(x)$

Taylor's theorem: Suppose that $f(x)$ has $n+1$ derivatives on (a,b), and $x_0 \in (a,b)$. Then for each $x \in (a,b)$, there exists ξ, depending on x, lying strictly between x and x_0 such that

$$f(x) = f(x_0) + \sum_{j=1}^{n} \left(\frac{f^{(j)}(x_0)}{j!}(x - x_0)^j \right) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}.$$

← Remainder or error term

n^{th} **Taylor polynomial:** $T_n(x) = f(x_0) + \sum_{j=1}^{n} \left(\frac{f^{(j)}(x_0)}{j!}(x - x_0)^j \right)$. ← Taylor Series Expansion

Maclaurin polynomial: A Taylor polynomial expanded about $x_0 = 0$ is also called a Maclaurin polynomial.

Remainder term: $R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}$ is precisely $-(T_n(x) - f(x))$.

Error term: Another name for the remainder term.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

Taylor Series Polynomial

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}$$

Error Analysis of Taylor Series

- Note that $f(x) = T_n(x) + R_n(x)$
- Absolute error is $|f(x) - T_n(x)| = |R_n(x)|$
- Absolute error depends on three factors:
  - $|x - x_0|^{n+1}$
  - $\frac{1}{(n+1)!}$
  - $\left|f^{(n+1)}(\xi)\right|$
- An error bound can be found by finding an upper bound on $\left|f^{(n+1)}(\xi)\right|$.

for sin & cos

what is limit?

$\left| f^{(n+1)}(x) \right| < 1$

Error Analysis for Exponential Example

```
# Cell 3
# error analysis of Taylor Series approximation of
e^8
# assumes cell two has been run
er=0.0*k
for i in np.nditer(k):
    er[i]=math.exp(8)-y[i]
fig, ax = plt.subplots()
ax.plot(k, er)
ax.set(xlabel='k', ylabel='error',
       title='Taylor Series Approximation')
plt.show()
```

n^{th} **Taylor polynomial:** $T_n(x) = f(x_0) + \sum_{j=1}^{n} \left(\frac{f^{(j)}(x_0)}{j!} (x - x_0)^j \right)$

$f(x) = \sin(x) \qquad x_0 = 0$

1st order $\rightarrow j = 1$

$T_1(x) = f(x_0) + \sum_{j=1}^{1} \left(\frac{f^{(1)}(x_0)}{1!} (x - x_0)^1 \right)$

$= \sin(0) + \frac{\frac{d}{dx}\sin(0)}{1!} (x - 0)^1)$

$\frac{d}{dx} \sin(0) = 0 \qquad = 0 + 0(x - 0) = 0$

$T_2 \rightarrow j = 2 \quad T_2(x) = f(x_0) + \sum_{j=1}^{2} \left(\frac{f^{(j)}(x_0)}{j!} (x - x_0)^j \right)$

Error for sine

```
# cell 4
# Demonstration of T1, T2 and T3 for sine
def T1(x):
    return x
def T2(x):
    return x - x**3/6.0
def T3(u):
    return T2(u)+u**5/120.0
x=np.linspace(-math.pi,math.pi,101)
t1=0.0*x
t2=0.0*x
t3=0.0*x
sin_x=0.0*x
for i in range(0,len(x)):
    t1[i]=T1(x[i])
    t2[i]=T2(x[i])
    t3[i]=T3(x[i])
    sin_x[i]=math.sin(x[i])
fig, ax = plt.subplots()
ax.plot(x, sin_x,label="sine(x)")
ax.plot(x,t1,label="T1")
ax.plot(x,t2,label="T2")
ax.plot(x,t3,label="T3")
ax.set(xlabel='x', ylabel='f(x)',
       title='Taylor Series Approximation of sine')
ax.legend()
plt.show()
```

Derivative Calculator

- Very helpful site
- Do not become dependent on it; it won't be available for the test
- Derivative Calculator Website

[1]: Cheney and Kincaid (2004), *Numerical Mathematics and Computing, 5th edition*.