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Wednesday, October 16, 2024 2:00 PM

Section 1

MANE 3351

Lecture 15

Classroom Management

Agenda

- Romberg Integration
- Lab 7, if not completed
- Homework 4 due Monday

Calendar

Date	Lecture	Lab
10/14	Simpson's Rule, Homework 4	Lab 7
10/16	Romberg Integration	Lab 7, continued
10/21	No Class, Dr. Timmer on ABET Visit	No Class, Dr. Timmer on ABET Visit
10/23	Gaussian Quadrature, Homework 5	Lab 8
10/28	Numerical Differentiation (not on Test 2)	Lab 8, continued
11/4	Numerical Integration (not on Test 2)	Lab 9
11/6	Linear Algebra, part 1 (not on Test 2)	Lab 9, continued
11/11	**Test 2 (Root Finding**	No Lab

Resources

Handouts

- Lecture 15 Slides
- Lecture 15 Marked Slides

Lecture 15

Today's topic is Romberg Integration

- Clever combination of trapezoid rule and Richardson's Extrapolation
- Highly accurate
- Cheney and Kincaid (2004)[a] show example output in the form of a lower triangle from Romberg integration

[a] Cheny, W., and Kincaid, D., (2004), *Numerical Mathematics and Computer, 5th edition*

The *Romberg algorithm* produces a triangular array of numbers, all of which are numerical estimates of the definite integral $\int_a^b f(x)\,dx$. The array is denoted here by the notation

1st step

1st guess, Trapezoid Rule

$R(1,0) = f(R(0,0))$

$R(1,1) = f(R(1,0), R(0,0))$

$$
\begin{array}{cccccc}
R(0,0) & & & & & \\
R(1,0) & R(1,1) & & & & \\
R(2,0) & R(2,1) & R(2,2) & & & \\
R(3,0) & R(3,1) & R(3,2) & R(3,3) & & \\
\vdots & \vdots & \vdots & \vdots & \ddots & \\
R(n,0) & R(n,1) & R(n,2) & R(n,3) & \cdots & R(n,n)
\end{array}
$$

final

Figure 1: Romberg Integration Results

Step 1

The first step is to calculate $R(0,0)$

- $R(0,0)$ is the result of applying the Trapezoid rule with 1 interval
- $R(0,0) = \frac{1}{2}(b-a)\left[f(a) + f(b)\right]$
- For our example of the standard normal pdf with $a = -5$, and $b = 0$, we observe
  - $R(0,0) = \frac{1}{2}(0-(-5.0))\left[f(-5.0) + f(0.0)\right] = \frac{1}{2}(5.0)\left[0.0 + 0.398942\right] = 0.997355$
  - This is a very poor approximation to the true value of 0.5

Step 2

Start a second row and calculate $R(1,0)$. For each new row, double the number of intervals used in the trapezoid rule

- $R(1,0)$ is the trapezoid with two intervals
- The general formula for $R(n,0)$ is

$n = 2 \quad \sum_{k=1}^{2^{n-1}} \quad \sum_{1}^{2^{2-1}} = \sum_{1}^{2}$

$$R(n,0) = \frac{1}{2} R\left(n-1,0\right) + h \sum_{k=1}^{2^{n-1}} f\left[a + \left(2k-1\right)h\right]$$

where $h = \left(b-a\right)/2^{n}$ and $n \geq 1$

Step 3

Complete the second row and calculate $R(1,1)$

- The calculation of $R(1,1)$ utilizes Richardson's extrapolation, $R(1,1) = f\left[R(1,0), R(0,0)\right]$
- The general formula for $R(n,m)$ is

$$R(n,m) = R(n,m-1) + \frac{1}{4^m - 1}\left[R(n,m-1) - R(n-1,m-1)\right]$$

Error Analysis

- Cheney and Kincaid (2004)[a] reports the following errors
  - The error for the first column is $\mathcal{O}(h^2)$
  - The error for the second column is $\mathcal{O}(h^4)$
  - The error for the third column is $\mathcal{O}(h^8)$
  - and so on

[a]Cheny, W., and Kincaid, D., (2004), *Numerical Mathematics and Computer, 5th edition*

Pseudo-code

Cheney and Kincaid (2004)[a] provide the following pseudo-code

[redacted]

procedure *Romberg*$(f, a, b, n, (r_{ij}))$
real array $(r_{ij})_{0:n \times 0:n}$
integer i, j, k, n

real a, b, h, sum
interface external function f
$h \leftarrow b - a$
$r_{00} \leftarrow (h/2)[f(a) + f(b)]$
for $i = 1$ **to** n **do**
  $h \leftarrow h/2$
  $sum \leftarrow 0$
  for $k = 1$ **to** $2^i - 1$ **step** 2 **do**
    $sum \leftarrow sum + f(a + kh)$
  end for
  $r_{i0} \leftarrow \frac{1}{2} r_{i-1,0} + (sum)h$
  for $j = 1$ **to** i **do**
    $r_{ij} \leftarrow r_{i,j-1} + (r_{i,j-1} - r_{i-1,j-1})/(4^j - 1)$
  end for

Python Code for Romberg Integration

```
import math
import numpy as np
def f(z):
    return (math.exp(-0.5*z**2)/((2.0*math.pi)**0.5))
#
a=float(input("Enter the lower limit of the integral: "))
b=float(input("Enter the upper limit of the integral: "))
n=int(input("enter the number of interations (n): "))
#
# initialize matrix r
r=np.zeros(shape=(n+1,n+1))
h=b-a
#find R(0,0)
r[0][0]=(h/2.0)*(f(a)+f(b))
for i in range(1,n+1):
    h=h/2.0
    sum=0.0
```

Romberg Integration by Hand

Wednesday, October 16, 2024 2:21 PM

$a = -5, \quad b = 0 \qquad b - a = h$

$$R(0,0) = \frac{1}{2}(b-a)\left[f(a) + f(b)\right]$$

$$= \frac{1}{2}(0 - -5)\left[f(-5) + f(0)\right]$$

$$= \frac{1}{2}(5)\left[1.487 \times 10^{-6} + .3989\right]$$

$$= \frac{1}{2}(5)(.3989)$$

$$= .99725$$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

Std. normal pdf

$a = -5$
$b = 0$

find $R(1,0)$

Wednesday, October 16, 2024 2:31 PM

$h = b - a / 2^n = \frac{0 - -5}{2^1} = 2.5$

$R(1,0) = \frac{1}{2}R(0,0) + h \sum_{k=1}^{2^{1-1}} [f(a + (2k-1) h]$

$= \frac{1}{2}R(0,0) + h \sum_{k=1}^{1} f[a + (2k-1) h]$

$= \frac{1}{2}(.99736) + 2.5[f(-5 + (1)h)]$

$+ 2.5 [f(-2.5)]$

$= \frac{1}{2}(.99736) + 2.5(.017528) = \boxed{.5425}$

$R(1,1)$ $a=-5,\ b=0,\ h=2.5$
$n=1$

Wednesday, October 16, 2024 2:43 PM

$R(1,1) \to n=1, m=1$

$$R(n,m) = R(n,m-1) + \frac{1}{4^m-1}\left[R(n,m-1) - R(n-1,m-1)\right]$$

$$R(1,1) = R(1,0) + \frac{1}{4^1-1}\left[R(1,0) - R(0,0)\right]$$

$$= \cancel{.44}\ .54250 + \frac{1}{3}\left[.54250 - .99736\right]$$

$$= .39088$$