

MANE 3351

Section 1

MANE 3351

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Lecture 7

Classroom Management

Agenda

- Taylor Series Expansion
- Homework 2
- Schedule
- Lab session at 3:30

→ last material for
Part One

Resources

Handouts

- Lecture 7 Slides
- Lecture 7 Marked Slides

Assignments

- Homework 1 (assigned 9/16/2024, due 9/23/2024 (before 11:59 pm))
- Homework 2 (assigned 9/18/2024, due 9/25/2024 (before 11:59 pm - no late submissions))
- Lab 3 (assigned 9/18/2024, due 9/25/2024 (before 2:00pm))
- Read textbook pages 1 - 16

Schedule

Lecture/Lab	Date	Topic
7	9/20 9/18	Taylor Series, Homework 2 (due 9/25 - no late work), Lab 3 (due 9/25)
8	9/23 9/23	Roots of Equations, bisection method (not on Test 1)
9	9/25 9/25	Bisection Method Error Analysis, False Position (not on Test 1)
10	9/30	Test 1 (lectures 1-7)

Lecture Content

- Taylor Series Expansion

Taylor Series

Introduction to Taylor Series

Cheney and Kincaid [1] provide some commonly used Taylor series.

Taylor Series

Familiar (and useful) examples of Taylor series are the following:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad (|x| < \infty) \quad (1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \quad (|x| < \infty) \quad (2)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \quad (|x| < \infty) \quad (3)$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{k=0}^{\infty} x^k \quad (|x| < 1) \quad (4)$$

Example

To find e^8 , recall $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

$$\textcircled{0} \quad e^8 = 1 = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \frac{x^0}{0!} = 1$$

$$\textcircled{1} \quad e^8 = 1 + x = 1 + 8$$

$$\textcircled{2} \quad e^8 = 1 + 8 + \frac{x^2}{2!} = 1 + 8 + \frac{64}{2}$$

$$\textcircled{3} \quad e^8 = 1 + 8 + \frac{64}{2} + \frac{x^3}{3!} = 1 + 8 + \frac{64}{2} + \frac{512}{6}$$

Python Code for Jupyter Notebook

```

import math
import numpy as np
import matplotlib.pyplot as plt
def eTaylor(x,k):
    y=0.0
    for i in range(k):
        #print(i)
        y=y+(x**i)/math.factorial(i)
        #print("i=",i," y=",y," i!=" ,math.factorial(i)," x^i="
    return y
k=np.arange(21)
print(k)
print(k[0])
y=0.0*k
for i in np.nditer(k):
    #print(i)
    y[i]=eTaylor(8,i)

```

use of final function → go over setting up Anaconda Python

$\int -x;$

Taylor Series Expansion about a Point

Taylor's theorem: Suppose that $f(x)$ has $n + 1$ derivatives on (a, b) , and $x_0 \in (a, b)$. Then for each $x \in (a, b)$, there exists ξ , depending on x , lying strictly between x and x_0 such that

$$f(x) = f(x_0) + \sum_{j=1}^n \left(\frac{f^{(j)}(x_0)}{j!} (x - x_0)^j \right) + \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)^{n+1}.$$

Remainder
or
error

n^{th} Taylor polynomial: $T_n(x) = f(x_0) + \sum_{j=1}^n \left(\frac{f^{(j)}(x_0)}{j!} (x - x_0)^j \right).$

Maclaurin polynomial: A Taylor polynomial expanded about $x_0 = 0$ is also called a Maclaurin polynomial.

Remainder term: $R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)^{n+1}$ is precisely $-(T_n(x) - f(x))$.

Error term: Another name for the remainder term.

Figure 2: Taylor Series Polynomial

Source: textbook, page 14

Error Analysis of Taylor Series

- Note that $f(x) = T_n(x) + R_n(x)$
- Absolute error is $|f(x) - T_n(x)| = |R_n(x)|$
- Absolute error depends on three factors:
 - $|x - x_0|^{n+1}$
 - $\frac{1}{(n+1)!}$
 - $|f^{(n+1)}(\xi)|$
- An error bound can be found by finding an upper bound on $|f^{(n+1)}(\xi)|$.

sin or cos

Error Analysis for Exponential Example

```
# Cell 3  
# error analysis of Taylor Series approximation of e^8  
# assumes cell two has been run  
er=0.0*k  
for i in np.nditer(k):  
    er[i]=math.exp(8)-y[i]  
fig, ax = plt.subplots()  
ax.plot(k, er)  
ax.set(xlabel='k', ylabel='error',  
       title='Taylor Series Approximation')  
plt.show()
```

Error for sine

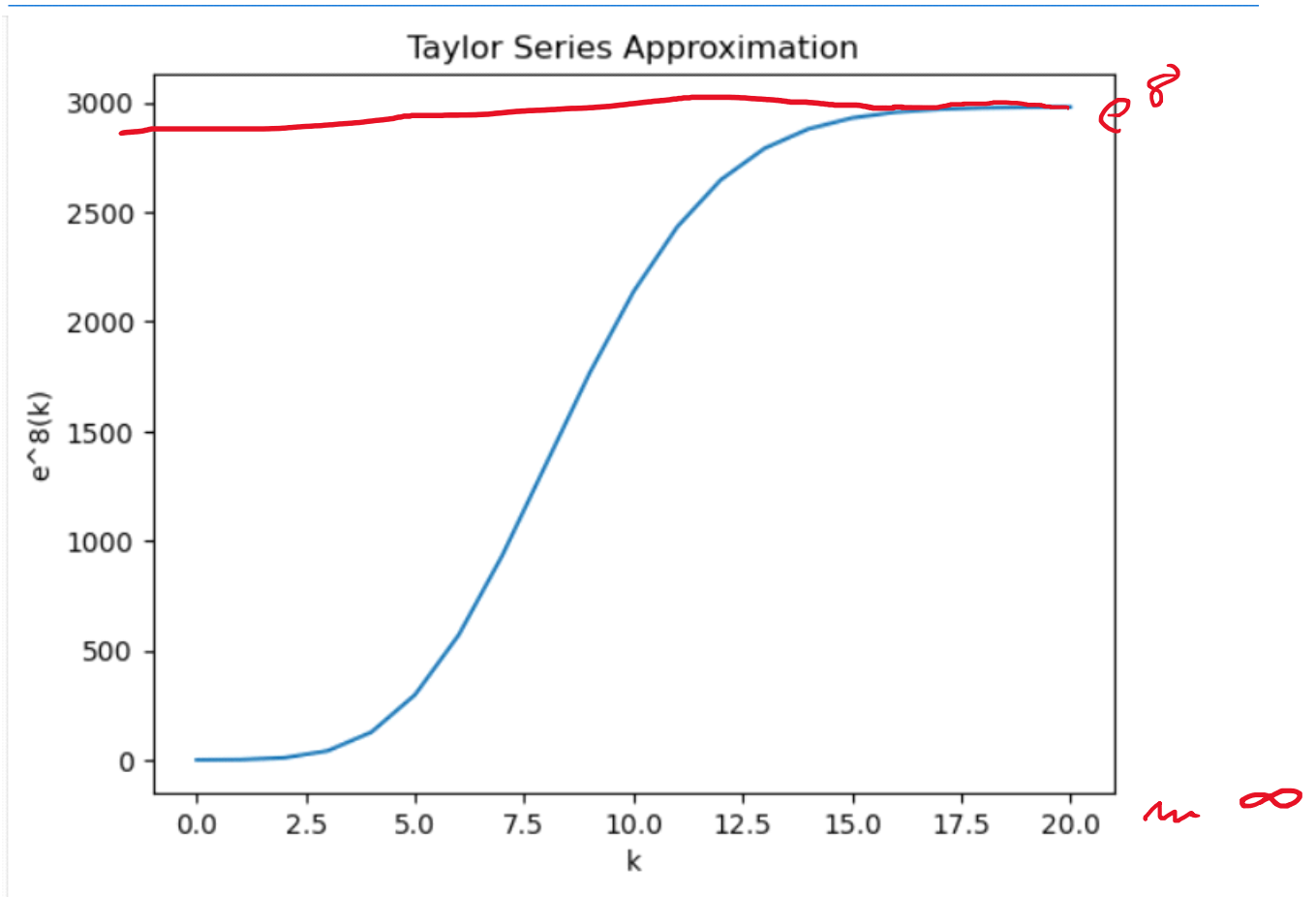
```
# cell 4
# Demonstration of T1, T2 and T3 for sine
def T1(x):
    return x
def T2(x):
    return x - x**3/6.0
def T3(u):
    return T2(u)+u**5/120.0
x=np.linspace(-math.pi,math.pi,101)
t1=0.0*x
t2=0.0*x
t3=0.0*x
sin_x=0.0*x
for i in range(0,len(x)):
    t1[i]=T1(x[i])
    t2[i]=T2(x[i])
    t3[i]=T3(x[i])
```

Derivative Calculator

- Very helpful site
- Do not become dependent on it; it won't be available for the test
- Derivative Calculator Website

[

1]: Cheney and Kincaid (2004), *Numerical Mathematics and Computing*, 5th edition.



Screen clipping taken: 9/18/2024 2:38 PM

$$e^x \quad T_3(x): f(x) = f(x_0) + \sum_{j=1}^3 \left(\frac{f^{(j)}(x_0)}{j!} (x-x_0)^j \right) \quad x_0 = 0$$

$$f^{(1)}(x) = \frac{d}{dx} [e^x] = e^x$$

$$f^{(2)}(x) = \frac{d}{dx} \left[\frac{d}{dx} (e^x) \right] = \frac{d}{dx} [f^{(1)}(x)] = e^x$$

$$f^{(3)}(x) = e^x$$

$$T_3(x) = e^0 + \frac{e^0}{1!} (x-0)' + \frac{e^0}{2!} (x-0)^2 + \frac{e^0}{3!} (x-0)^3$$

$$= 1 + \frac{1}{1!} (x) + \frac{1}{2!} (x)^2 + \frac{1}{3!} (x)^3$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$T_2(x)$ for $f(x) = e^x$ at $x_0 = 8$

$$\begin{aligned} T_2(x) &= f(x_0) + \sum_{j=1}^2 \left(\frac{f^{(j)}(x_0)}{j!} (x-x_0)^j \right) \\ &= e^8 + \frac{e^8}{1!} (x-8) + \frac{e^8}{2!} (x-8)^2 \end{aligned}$$