Section 1

MANE 3351

Lecture 15

Classroom Management

Agenda

- Romberg Integration
- Lab 7, if not completed
- Homework 4 due Monday

Calendar

Date	Lecture	Lab
10/14	Simpson's Rule, Homework 4	Lab 7
10/16	Romberg Integration	Lab 7, continued
10/21	No Class, Dr. Timmer on ABET Visit	No Class, Dr. Timmer on ABET Visit
10/23	Gaussian Quadrature, Homework 5	Lab 8
10/28	Numerical Differentiation (not on Test 2)	Lab 8, continued
11/4	Numerical Integration (not on Test 2)	Lab 9
11/6	Linear Algebra, part 1 (not on Test 2)	Lab 9, continued
11/11	**Test 2 (Root Finding**	No Lab

Resources

Handouts

- Lecture 15 Slides
- Lecture 15 Marked Slides

Lecture 15

Today's topic is Romberg Integration

- Clever combination of trapezoid rule and Richardson's Extrapolation
- Highly accurate
- Cheney and Kincaid (2004)[a] show example output in the form of a lower triangle from Romberg integration

[a]Cheny, W., and Kincaid, D., (2004), *Numerical Mathematics and Computer, 5th edition*

The *Romberg algorithm* produces a triangular array of numbers, all of which are numerical estimates of the definite integral $\int_a^b f(x)\,dx$. The array is denoted here by the notation

$$\begin{array}{llllll}
R(0,0) & & & & & \\
R(1,0) & R(1,1) & & & & \\
R(2,0) & R(2,1) & R(2,2) & & & \\
R(3,0) & R(3,1) & R(3,2) & R(3,3) & & \\
\vdots & \vdots & \vdots & \vdots & \ddots & \\
R(n,0) & R(n,1) & R(n,2) & R(n,3) & \cdots & R(n,n)
\end{array}$$

Figure 1: Romberg Integration Results

Step 1

The first step is to calculate $R(0,0)$

- $R(0,0)$ is the result of applying the Trapezoid rule with 1 interval
- $R(0,0) = \frac{1}{2}(b-a)\left[f(a) + f(b)\right]$
- For our example of the standard normal pdf with $a = -5$, and $b = 0$, we observe
  - $R(0,0) = \frac{1}{2}(0-(-5.0))\left[f(-5.0) + f(0.0)\right] = \frac{1}{2}(5.0)\left[0.0 + 0.398942\right] = 0.997355$
  - This is a very poor approximation to the true value of 0.5

Step 2

Start a second row and calculate $R(1,0)$. For each new row, double the number of intervals used in the trapezoid rule

- $R(1,0)$ is the trapezoid with two intervals
- The general formula for $R(n,0)$ is

$$R(n,0) = \frac{1}{2}R\left(n-1,0\right) + h\sum_{k=1}^{2^{n-1}} f\left[a + \left(2k-1\right)h\right]$$

where $h = \left(b-a\right)/2^n$ and $n \geq 1$

Step 3

Complete the second row and calculate $R(1,1)$

- The calculation of $R(1,1)$ utilizes Richardson's extrapolation, $R(1,1) = f\left[R(1,0), R(0,0)\right]$
- The general formula for $R(n,m)$ is

$$R(n,m) = R(n,m-1) + \frac{1}{4^m - 1}\left[R(n,m-1) - R(n-1,m-1)\right]$$

Error Analysis

- Cheney and Kincaid (2004)[a] reports the following errors
  - The error for the first column is $\mathcal{O}(h^2)$
  - The error for the second column is $\mathcal{O}(h^4)$
  - The error for the third column is $\mathcal{O}(h^8)$
  - and so on

[a] Cheny, W., and Kincaid, D., (2004), *Numerical Mathematics and Computer, 5th edition*

Pseudo-code

Cheney and Kincaid (2004)[a] provide the following pseudo-code

[redacted]

```
procedure Romberg(f, a, b, n, (r_ij))
real array (r_ij)_{0:n×0:n}
integer i, j, k, n

real a, b, h, sum
interface external function f
h ← b − a
r_00 ← (h/2)[f(a) + f(b)]
for i = 1 to n do
    h ← h/2
    sum ← 0
    for k = 1 to 2^i − 1 step 2 do
        sum ← sum + f(a + kh)
    end for
    r_i0 ← (1/2)r_{i−1,0} + (sum)h
    for j = 1 to i do
        r_ij ← r_{i,j−1} + (r_{i,j−1} − r_{i−1,j−1})/(4^j − 1)
    end for
```

Python Code for Romberg Integration

```
import math
import numpy as np
def f(z):
    return (math.exp(-0.5*z**2)/((2.0*math.pi)**0.5))
#
a=float(input("Enter the lower limit of the integral: "))
b=float(input("Enter the upper limit of the integral: "))
n=int(input("enter the number of interations (n): "))
#
# initialize matrix r
r=np.zeros(shape=(n+1,n+1))
h=b-a
#find R(0,0)
r[0][0]=(h/2.0)*(f(a)+f(b))
for i in range(1,n+1):
    h=h/2.0
    sum=0.0
```

Romberg Integration by Hand