Section 1

MANE 3351

Lecture 7

Classroom Management

Agenda

- Taylor Series Expansion
- Homework 2
- Schedule
- Lab session at 3:30

Resources

Handouts

- Lecture 7 Slides
- Lecture 7 Marked Slides

Assignments

- Homework 1 (assigned 9/16/2024, due 9/23/2024 (before 11:59 pm)
- Homework 2 (assigned 9/18/2024, due 9/25/2024 (before 11:59 pm - no late submissions)
- Lab 3 (assigned 9/18/2024, due 9/25/2024 (before 2:00pm))
- Read textbook pages 1 - 16

Schedule

Lecture/Lab	Date	Topic
7	9/20	Taylor Series, Homework 2 (due 9/25 - no late work), Lab 3 (due 9/25)
8	9/23	Roots of Equations, bisection method (not on Test 1)
9	9/25	Bisection Method Error Analysis, False Position (not on Test 1)
10	9/30	Test 1 (lectures 1-7)

Lecture Content

- Taylor Series Expansion

Taylor Series

Introduction to Taylor Series

Cheney and Kincaid [ˆ1] provide some commonly used Taylor series.

Taylor Series

Familiar (and useful) examples of Taylor series are the following:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad (|x| < \infty) \tag{1}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \qquad (|x| < \infty) \tag{2}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \qquad (|x| < \infty) \tag{3}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{k=0}^{\infty} x^k \qquad (|x| < 1) \tag{4}$$

Example

To find e^8, recall $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

0. $e^8 = 1$
1. $e^8 = 1 + x = 1 + 8$
2. $e^8 = 1 + 8 + \frac{x^2}{2!} = 1 + 8 + \frac{64}{2}$
3. $e^8 = 1 + 8 + \frac{64}{2} + \frac{x^3}{3!} = 1 + 8 + \frac{64}{2} + \frac{512}{6}$

Python Code for Jupyter Notebook

```
import math
import numpy as np
import matplotlib.pyplot as plt
def eTaylor(x,k):
    y=0.0
    for i in range(k):
        #print(i)
        y=y+(x**i)/math.factorial(i)
        #print("i=",i," y=",y," i!= ",math.factorial(i)," x^i=
    return y
k=np.arange(21)
print(k)
print(k[0])
y=0.0*k
for i in np.nditer(k):
    #print(i)
    y[i]=eTaylor(8,i)
```

Taylor Series Expansion about a Point

Taylor's theorem: Suppose that $f(x)$ has $n+1$ derivatives on (a,b), and $x_0 \in (a,b)$. Then for each $x \in (a,b)$, there exists ξ, depending on x, lying strictly between x and x_0 such that

$$f(x) = f(x_0) + \sum_{j=1}^{n} \left(\frac{f^{(j)}(x_0)}{j!}(x - x_0)^j \right) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}.$$

n^{th} **Taylor polynomial:** $T_n(x) = f(x_0) + \sum_{j=1}^{n} \left(\frac{f^{(j)}(x_0)}{j!}(x - x_0)^j \right)$.

Maclaurin polynomial: A Taylor polynomial expanded about $x_0 = 0$ is also called a Maclaurin polynomial.

Remainder term: $R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}$ is precisely $-(T_n(x) - f(x))$.

Error term: Another name for the remainder term.

Figure 2: Taylor Series Polynomial

Source: textbook, page 14

Error Analysis of Taylor Series

- Note that $f(x) = T_n(x) + R_n(x)$
- Absolute error is $|f(x) - T_n(x)| = |R_n(x)|$
- Absolute error depends on three factors:
  - $|x - x_0|^{n+1}$
  - $\frac{1}{(n+1)!}$
  - $|f^{(n+1)}(\xi)|$
- An error bound can be found by finding an upper bound on $|f^{(n+1)}(\xi)|$.

Error Analysis for Exponential Example

```
# Cell 3
# error analysis of Taylor Series approximation of e^8
# assumes cell two has been run
er=0.0*k
for i in np.nditer(k):
    er[i]=math.exp(8)-y[i]
fig, ax = plt.subplots()
ax.plot(k, er)
ax.set(xlabel='k', ylabel='error',
       title='Taylor Series Approximation')
plt.show()
```

Error for sine

```
# cell 4
# Demonstration of T1, T2 and T3 for sine
def T1(x):
    return x
def T2(x):
    return x - x**3/6.0
def T3(u):
    return T2(u)+u**5/120.0
x=np.linspace(-math.pi,math.pi,101)
t1=0.0*x
t2=0.0*x
t3=0.0*x
sin_x=0.0*x
for i in range(0,len(x)):
    t1[i]=T1(x[i])
    t2[i]=T2(x[i])
    t3[i]=T3(x[i])
```

Derivative Calculator

- Very helpful site
- Do not become dependent on it; it won't be available for the test
- Derivative Calculator Website

[

1]: Cheney and Kincaid (2004), *Numerical Mathematics and Computing, 5th edition*.